DIRECTIONS  This exam has two parts, Part A has 4 short answer problems (24 points) while Part B has 6 traditional problems (72 points). Closed book, no calculators – but you may use one 3” × 5” card with notes.

Part A: Proof or Counterexample (4 problems, 6 points each)
Here let \( f_n(x) \), \( n = 1, 2, \ldots \) be a sequence of continuous functions for \( 0 \leq x < \infty \) with \( f_n(x) = 0 \) for \( x \geq n \). For a counterexample, a clear sketch may be completely adequate.

A–1. If \( f_n(x) \) converges to zero for every \( x \in [0, 1] \), then \( f_n \) converges to zero uniformly on the interval \([0, 1]\).

A–2. If \( f_n(x) \) converges to zero for every \( x \in [0, 1] \), then \( \int_0^1 f_n(x) \, dx \to 0 \).

A–3. If \( f_n(x) \) converges to zero uniformly for \( x \) in the interval \( x \in [0, 1] \), then \( \int_0^1 f_n(x) \, dx \to 0 \).

A–4. \( f_n(x) \) converges to zero uniformly for \( x \) in the interval \( 0 \leq x < \infty \), then \( \int_0^\infty f_n(x) \, dx \to 0 \).

Part B: Traditional Problems (6 problems, 12 points each)

B–1. Let \( f \in C^2([0, 3]) \) have the properties \( f(0) = 4 \), \( f(1) = 3 \), and \( f(3) = 6 \). Show there is at least one point \( z \in [0, 3] \) where \( f''(z) \geq \text{const} > 0 \) and give an estimate for this constant.

B–2. Let \( f(x) \in C([0, 2]) \) be a continuous function with the property: \( \int_0^2 f(x)h(x) \, dx = 0 \) for every function \( h \in C([0, 2]) \) that is zero at the end points: \( h(0) = h(2) = 0 \). Show that \( f(x) \equiv 0 \).

B–3. Let \( f(x) \in C([0, 1]) \). Find \( \lim_{n \to \infty} n \int_0^1 f(x)e^{-2nx} \, dx \) (justify your assertions).

[continued on the next page]
B–4. Let $\varphi_k(x), \ x \in \mathbb{R}^2$, be a sequence of smooth functions with the following properties

i). $\varphi_k(x) \geq 0$ for $\|x\| < 1/k$, $\varphi_k(x) = 0$ for $\|x\| \geq 1/k$,

ii). $\int\int_{\mathbb{R}^2} \varphi_k(x) \, dx = 1$.

For a continuous function $f(x)$ with $f(x) = 0$ for $x$ outside a compact set $K$, define

$$f_k(x) := \int\int_{\mathbb{R}^2} f(y) \varphi_k(x - y) \, dy.$$

a) Show that $\lim_{n \to \infty} f_k(x) = f(x)$, and that this convergence is uniform.

B–5. Compute $\int\int_{\mathbb{R}^2} \frac{dx \, dy}{[4 + 5x^2 - 2xy + 2y^2]^{3/2}}$.

B–6. Let $x = (x_1, \ldots, x_n)$ and assume that $u(x)$ depends only on $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, so $u(x) = f(r)$ for some function $f$ depending only on $r$.

a) Show that $\frac{\partial u}{\partial x_i} = \frac{df}{dr} \frac{x_i}{r}$.

b) Show that $\frac{\partial^2 u}{\partial x_i^2} = \frac{d^2 f}{dr^2} \left( \frac{x_i^2}{r^2} \right) + \frac{df}{dr} \left( \frac{1}{r} - \frac{x_i^2}{r^3} \right)$.

c) Compute $\Delta u := \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$ in terms of $f$ and its derivatives.

d) If $n = 3$, use this to find all functions $u(x) = f(r)$ that satisfy $\Delta u = 0$ for all $x \neq 0$. 