1. Say one has a real normed linear space whose norm satisfies the parallelogram identity
\[ \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2. \]
Define the bilinear form \(B(x, y)\) by the rule
\[ B(x, y) := \frac{1}{4} \left[ \|x + y\|^2 - \|x - y\|^2 \right]. \]
Show that \(B(x, y)\) has all of the properties of an inner product including \(B(x, x) = \|x\|^2\).

[First observed by von Neumann]

MORAL: If a norm satisfies the parallelogram identity, then one can use it to define a compatible inner product. The parallelogram identity is both necessary and sufficient.

2. Consider \(\mathbb{R}^2\) with vectors \(X = (x_1, x_2)\) having the \(p\) norm, \(\|x\|_p := (|x_1|^p + |x_2|^p)^{1/p}, \quad p \geq 1\). Show that if \(p \neq 2\), this norm does not arise from an inner product.

3. Consider the Fourier series (formally, so we don’t yet worry about convergence)
\[ f(x) = \sum_{-\infty < k < \infty} c_k e^{ikx} \quad \text{where} \quad c_k \in \mathbb{C} \]  \hspace{1cm} (1)
with the complex inner product \(\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)} \, dx\).

a) Show that the functions \(e^{ikx}\) for integers \(k = 0, \pm 1, \pm 2, \ldots\) are mutually orthogonal.

b) Show that \(c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx\). The \(c_k\) are the Fourier coefficients of \(f\).

c) Assuming that equality holds in (1), show that formally
\[ \|f\|^2 = \int_{-\pi}^{\pi} |f(t)|^2 = 2\pi \sum_{-\infty < k < \infty} |c_k|^2. \]

This is a generalization of the Pythagorean theorem.

d) Define the linear map \(P_N\) by \((P_Nf)(x) := \sum_{|k| \leq N} c_k e^{ikx}\). Show that
\[ P_N^2 = P_N, \quad \text{that is,} \quad P_N(P_Nf) = P_Nf \quad \text{for any } f \]
and also that
\[ \langle P_Nf, g \rangle = \langle f, P_Ng \rangle \quad \text{for any } f, g \]
(the second property says that \(P_N\) is self-adjoint). These two properties are often summarized by saying that the map \(P_N\) is an orthogonal projection.
e) Show that \( \|f\|^2 = \|P_N f\|^2 + \|(I - P_N) f\|^2 \) and hence that \( \|P_N f\| \leq \|f\| \).

4. Let \( f(x) = x^2 \) for \( -\pi \leq x \leq \pi \).
   a) Find the Fourier series \( \sum_{-\infty}^{\infty} c_k e^{ikx} \) in \( L_2(-\pi, \pi) \) for \( f \).
   b) Use this to compute \( \sum_{k=1}^{\infty} \frac{1}{n^4} \).

5. In any vector space \( V \) with an inner product, let \( \mathcal{W} \) be a subspace. We want to define the orthogonal projection of \( v \in V \) into \( \mathcal{W} \), written \( P_{\mathcal{W}} v \). One approach is to assume we have written \( v \) as 
   \[ v = v_1 + v_2, \quad \text{where} \quad v_1 \in \mathcal{W} \quad \text{and} \quad v_2 \perp \mathcal{W}. \]

Then we define the operator \( P_{\mathcal{W}} \) as \( P_{\mathcal{W}} v := v_1 \).

   a) Show that \( P_{\mathcal{W}} v \) is the point in \( \mathcal{W} \) that is closest to \( v \) by proving that for any \( w \in \mathcal{W} \)
   \[ \|v - w\|^2 = \|v - P_{\mathcal{W}} v\|^2 + \|P_{\mathcal{W}} v - w\|^2. \]

   b) As an application of this, let \( \mathcal{T}_N \) be the subspace of trigonometric polynomials of degree at most \( N \), that is, functions of the form \( \sum_{|k| \leq N} c_k e^{ikx} \). For short, write \( P_N := P_{\mathcal{T}_N} \). Given a function \( f \), show that for any function \( g \in \mathcal{T}_N \) one has \( \|f - P_N f\| \leq \|f - g\| \). Thus, in this norm the Fourier projection \( P_{\mathcal{T}_N} f \) is closer to \( f \) than any other function in \( \mathcal{T}_N \).

   c) Let \( D = d/dx \). Show that \( P_N D = DP_N \), that is, \( P_N(Df) = D(P_Nf) \) for all continuously differentiable \( 2\pi \)-periodic functions \( f \).

[Last revised: April 5, 2005]