1. a) Let \( S \subset \mathbb{R}^n \) be a subspace and \( Z \in \mathbb{R}^n \) a given vector. Find a unit vector \( X \) that is perpendicular to \( S \) with \( \langle X, Z \rangle \) as large as possible.

b) Compute \( \min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \, dx \).

c) Compute \( \max \int_{-1}^{1} x^3 h(x) \, dx \) where \( h \in L^2(-1,1) \) is subject to the restrictions

\[
\int_{-1}^{1} h(x) \, dx = \int_{-1}^{1} xh(x) \, dx = \int_{-1}^{1} x^2 h(x) \, dx = 0; \quad \int_{-1}^{1} |h(x)|^2 \, dx = 1.
\]

2. Let \( \alpha \) be an irrational real number and let \( f(\theta) \) be a continuous \( 2\pi \) periodic function, \( 0 \leq \theta \leq 2\pi \). Prove that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(2\pi k \alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \, d\theta.
\]

3. Let \( f(x) \) be a continuous real-valued function with period \( 2\pi \), so \( f(x + 2\pi) = f(x) \) for all real \( x \). If also for some irrational \( \alpha \in \mathbb{R} \) we know that \( f(x + 2\pi \alpha) = f(x) \) for all real \( x \), show that \( f(x) \equiv \) constant.

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