
1. Let \( \gamma : \mathbb{R} \to \mathbb{R}^3 \) describe a smooth curve and let \( V \in \mathbb{R}^3, V \neq 0 \). be a fixed vector. Assume \( \gamma'(t) \perp V \) for all \( t \).
   a) If \( \gamma(0) \perp V \), show that \( \gamma(t) \perp V \) for all \( t \).
   b) Even if \( \gamma(0) \) is not perpendicular to \( V \), show that \( \gamma(t) \) lies in a two-dimensional plane and find the equation of this plane.

2. The curve \( y = x^{2/3} \) \(-\infty < x < \infty\) has a cusp at the origin (as you can see from a sketch). Find smooth \( (C^\infty) \) functions \( x(t), y(t), -\infty < t < \infty \) that parameterize this curve. [The point of this is that even though a curve \( \gamma(t) = (x(t), y(t)) \) may have a parameterization by smooth functions \( x(t), y(t) \), we might not want to say that the curve is smooth.]

3. Find a smooth function \( f(x) \), \( x \in \mathbb{R} \) with the properties: \( f(x) = 1 \) for \( |x| \leq 1 \), \( f(x) = 0 \) for \( |x| \geq 2 \).

4. a) Let \( y = f(x) \) define a smooth curve in the plane. If \( P, Q \) and \( R \) are three distinct points on the curve, let \( \Gamma_{PQR} \) be the circle that passes through these three points (we allow that \( \Gamma_{PQR} \) might be a straight line, which can be viewed as a circle with infinite radius). In the limit as both \( Q \to P \) and \( R \to P \) show that this circle \( \Gamma_P \) is tangent to the curve at \( P \) and that in addition the second derivative of the curve and the circle agree at \( P \). [If the circle \( \Gamma_P \) has radius \( R \), we say that the curvature of \( y = f(x) \) at \( P \) is \( 1/R \).]
   b) Use this to obtain a formula for the curvature in terms of \( f \), \( f' \), and \( f'' \).

5. The \( n^{th} \) Legendre polynomial is \( P_n(x) = \frac{d^n}{dx^n}(x^2 - 1)^n \).
   a) Show that \( P_n(x) \) is a polynomial of degree \( n \).
   b) Show that \( P_n(x) \) has exactly \( n \) real distinct zeroes in the interval \( \{-1 < x < 1\} \).

6. Let \( A(t) = (a_{ij}(t)) \) be a square matrix of real numbers whose entries are smooth functions of \( t \in \mathbb{R} \) and assume that \( A(0) \) is invertible.
   a) Find a formula for the derivative of the inverse matrix, \( A^{-1} (t) \) in terms of \( A(0) \) and \( A'(0) \). [SUGGESTION: begin from \( A(t)A^{-1}(t) = I \).]
b) Recall that the trace of a square matrix is the sum of its diagonal elements. Show that
\[ \frac{d}{dt} \det(A(t))|_{t=0} = [\det(A(0))] \text{trace} \left( A^{-1}(0)A'(0) \right). \]

[SUGGESTION: First do the special case where \( A(0) = I \) and then reduce the general case to this special case.]

**Bonus Problem:** Let \( f(x, y) : \mathbb{R}^2 \to \mathbb{R} \) be a smooth function with exactly one critical point, and that critical point is a strict local minimum (say the critical point is at the origin and \( f''(0, 0) = I \)). Can one conclude that the origin is the *global* minimum? Proof or counterexample.