1. (Euler) Let \( f(x) \) be a differentiable function of \( x := (x_1, x_2, x_3) \) for all \( x \in \mathbb{R}^3 \). If \( f \) is homogeneous of degree \( k \) in the sense that \( f(cx) = c^k f(x) \) for all \( c > 0 \), show that \( x \cdot \nabla f(x) = k f(x) \).

2. Let \( B \subset \mathbb{R}^2 \) be the rectangle \( 0 \leq x \leq 2, 0 \leq y \leq 1 \). Show that
\[
\int_{B} \left( x^2 + 2y^3 \right) \sqrt{2 + \sin(xy)} \, dx \, dy \leq 2 + \sqrt{7}.
\]
[SUGGESTION: Split \( B \) into two smaller regions.]

3. Evaluate \( \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dx \, dy \). Use this to compute \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \).

4. Let \( v(x,t) := \int_{2x-t}^{2x+t} g(s) \, ds \), where \( g \) is a continuous function. Show that \( v_{xx} = 4v_{tt} \).

5. Let \( f(x) = x \sin(\pi/x) \) for \( x \neq 0 \) and \( f(0) = 0 \). Draw a sketch of the curve \( y = f(x) \) for \( 0 \leq x \leq \pi \). Using straight line segment approximations to the arc length, show that this curve has infinite arc length. [As a first step, show geometrically that the length of the portion (one arch) of the curve for \( \frac{1}{n+1} \leq x \leq \frac{1}{n} \) is at least \( 2/(n + \frac{1}{2}) \).]

This is the standard example of a non-rectifiable curve.

6. a) Compute
\[
\iint_{\mathbb{R}^2} \frac{dxdy}{(1 + 4x^2 + 9y^2)^2}, \quad \iint_{\mathbb{R}^2} \frac{dxdy}{(5 + x^2 + 2x + 9y^2)^2}, \quad \iint_{\mathbb{R}^2} \frac{dxdy}{(1 + 5x^2 - 4xy + 5y^2)^2}.
\]

b) Let \( h(t) \) be a given function and say you know that \( \int_{0}^{\infty} h(t) \, dt = \alpha \). If \( C \) be a positive definite \( 2 \times 2 \) matrix. Show that
\[
\iint_{\mathbb{R}^2} h(\langle x, Cx \rangle) \, dA = \frac{\pi \alpha}{\sqrt{\det C}}.
\]
c) Compute \( \int \int_{\mathbb{R}^2} e^{-(5x^2 - 4xy + 5y^2)} \, dx \, dy \).

d) Compute \( \int \int_{\mathbb{R}^2} e^{-(5x^2 - 4xy + 5y^2 - 2x + 3)} \, dx \, dy \).

e) Generalize part b) to obtain a formula for
\[
\int \int_{\mathbb{R}^n} h(\langle x, Cx \rangle) \, dV,
\]
where now \( C \) be a positive definite \( n \times n \) matrix. The answer will involve some integral involving \( h \) and also the “area” of the unit sphere \( S^{n-1} \hookrightarrow \mathbb{R}^n \).

7. A standard ingredient in many problems involves the eigenvalues \( \lambda \) and corresponding eigenfunctions \( u \) of the Laplacian \( \Delta = \nabla \cdot \nabla \), so
\[
-\Delta u = \lambda u \quad \text{in} \quad D \quad \text{with} \quad u = 0 \quad \text{on} \quad \partial B,
\]
Here \( D \) in \( \mathbb{R}^2 \) is a bounded region with boundary \( \partial B \). As usual, to be useful one wants numbers \( \lambda \) so that there is a solution \( u \) other than the trivial solution \( u \equiv 0 \). Show that
\[
\lambda = \frac{\int \int_D |\nabla u|^2 \, dA}{\int \int_D u^2 \, dA}.
\]
In particular, deduce that \( \lambda > 0 \).

8. a) Show that for any smooth function \( u(x,y) \)
\[
\int \int_D \Delta \phi \, dx \, dy = \int_{\partial D} \frac{\partial \phi}{\partial N} \, ds
\]
where \( \partial \phi/\partial N := \nabla \phi \cdot N \) is the outer normal directional derivative on \( \partial D \).

b) Let \( u(x,y,t) \) be a solution of the heat equation \( u_t = \Delta u \) for \( (x,y) \) in \( D \). Assume that the boundary, \( \partial D \), is insulated, so the outer normal derivative there is zero: \( \frac{\partial u}{\partial N} = 0 \) on \( \partial D \).

Show that \( Q(t) := \int \int_D u(x,y,t) \, dx \, dy \) is a constant.

9. Let \( f(x) \) be a continuous function for \( 0 \leq x \leq 1 \). Evaluate \( \lim_{n \to \infty} n \int_0^1 f(x)x^n \, dx \). (Justify your assertions.)
10. If the sequence \( \{a_n\} \) is bounded and \( c > 1 \), show that the series \( \sum_{n=1}^{\infty} \frac{a_n}{n^2} \) converges absolutely and uniformly for all complex \( z = x + iy \) in the closed half-plane \( c \leq x < \infty \).

**Bonus Problem** Let \( \gamma(t) \) be a closed piecewise smooth curve in the plane that encloses a convex region \( D \). We say that a curve \( \gamma_r \) is parallel to \( \gamma \) at distance \( r \) if every point on \( \gamma_r \) is outside \( D \) and has distance \( r \) from \( \gamma \). For instance, concentric circles are parallel. Discover a formula relating the arc length of \( \gamma_r \) to \( \gamma \). [SUGGESTION: One approach begins by looking at the cases where \( \gamma \) is a circle, a rectangle, and a (convex) polygon.]

**Bonus Problem** Make sense of the following: “Let \( \mathcal{D}_t \subset \mathbb{R}^2 \) be a family of bounded regions in the plane with smooth boundaries. These regions depend smoothly on a real parameter \( t \).” As a test of the effectiveness of your definition, use it to compute the derivative of

\[
J(t) := \iint_{\mathcal{D}_t} f(x,y) \, dx \, dy,
\]

at \( t = 0 \) where \( f(x,y) \) is a given smooth function and \( \mathcal{D}_t \) is the family of ellipsoids \( x^2 + (1+3t)y^2 = 1 \).