Directions

This exam has two parts, Part A has 2 short answer problems (5 points each) while Part B has 5 traditional problems (10 points each). Closed book, no calculators – but you may use one 3" × 5" card with notes.

Part A: Proof or Counterexample (2 problems, 5 points each)

Here let \( f_n(x), \ n = 1, 2, \ldots \) be a sequence of continuous functions for \( 0 \leq x \leq 2 \). For a counterexample, a clear sketch may be completely adequate.

A–1. If \( f_n(x) \) converges to zero for every \( x \in [0, 2] \), then \( f_n \) converges to zero uniformly on the interval \([0, 2]\).

A–2. If \( f_n(x) \) converges uniformly to zero for \( x \) in the interval \([0, 2]\), then \( \int_0^2 f_n(x) \, dx \to 0 \).

Part B: Traditional Problems (5 problems, 10 points each)

B–1. Compute \( \int_0^a x^2 \, dx \) (where \( 0 < a < \infty \)) directly by using Riemann sums (not as the anti-derivative). I suggest partitioning the interval \( 0 \leq x \leq a \) into segments having equal length. You may use without proof that \( 1^2 + 2^2 + \cdots + k^2 = \frac{1}{6} k(k + 1)(2k + 1) \).

B–2. Let \( a_n \) be a bounded sequence of real numbers. If \( c > 1 \), show that the series \( \sum_{n=1}^{\infty} \frac{a_n}{n^c} \) converges uniformly for \( x \geq c \).

B–3. Let \( f(x) \in C([0, 1]) \) be a continuous function with the property: \( \int_0^1 f(x)p(x) \, dx = 0 \) for every polynomial \( p(x) \). Show that \( f(x) \equiv 0 \).

B–4. Let
\[ p(x) := (x - 1)(x - 2)(x - 3)(x - 4)(x - 5)(x - 6) = x^6 - 21x^5 + \cdots \]
Clearly \( p(4) = 0 \). Denote by \( p(x, t) \) the polynomial obtained by replacing \(-21x^5\) by \((-21+t)x^5\), with \( |t| \) small. Let \( x(t) \) denote the perturbed value of root \( x = 4 \), so \( x(0) = 4 \).

a) Show that \( x(t) \) is a smooth function of \( t \) for all \( |t| \) sufficiently small.
b) Compute the sensitivity of this root as one changes \( t \), that is, compute \( dx(t)/dt \bigg|_{t=0} \).

B–5. Let \( f(x) \) and \( h(x, y) \) be continuous for \( x \) and \( y \) in the interval \([0, 2]\). Show that if \( \lambda > 0 \) is sufficiently small, the equation
\[ u(x) = f(x) + \lambda \int_0^2 h(x, y)u(y) \, dy \]
has a unique solution (that is, a solution exists and is unique).