DIRECTIONS. This exam has three parts, Part A has 4 shorter problems (5 points each), Part B has 5 traditional problems (10 points each). Closed book, no calculators – but you may use one 3” × 5” card with notes.

Part A: Shorter Problems (4 problems, 5 points each).

A–1. Give an example of a sequence of continuous functions $f_n(x), 0 \leq x \leq 1$, with $f_n(x) \to 0$ (pointwise) for all $x \in [0,1]$, but $\int_0^1 |f_n(x)| \, dx \to \infty$. A sketch is adequate.

A–2. In $L_2(-1,1)$ with the standard inner product, show that any even function is orthogonal to any odd function (of course assume that the functions are integrable).

A–3. Prove that the series $\sum_{k=1}^{\infty} \frac{(-1)^k \sin kx}{1 + k^2}$ converges absolutely and uniformly for all real $x$.

A–4. Let $u(x, y, t)$ be a solution of the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ for $(x, y)$ in a bounded domain $D \subset \mathbb{R}^2$ with the outer normal derivative $\nabla u \cdot N = 0$ on the boundary of $D$ (here $N$ is the unit outer normal vector field on the boundary).

If $Q(t) := \int\int_D u(x, y, t) \, dx \, dy$, show that $\frac{dQ}{dt} = 0$ and hence that $Q(t) = Q(0)$.

Part B: Traditional Problems (5 problems, 10 points each)

B–1. The following equations define a map $F : (x, y, z) \mapsto (u, v, w)$:

$$
\begin{align*}
  u(x, y, z) &= x + xyz^2 \\
  v(x, y, z) &= xz^2 + y \\
  w(x, y, z) &= 2x + cz + z^3
\end{align*}
$$

Clearly $F : (1,1,0) \mapsto (1,1,2)$. Write $p = (1,1,0)$ and $q = (1,1,2)$.

a) Compute the derivative $F'(p)$.

b) For which value(s) of the constant $c$ can the system of equations: can be solved for $x$, $y$, $z$ as smooth functions of $u$, $v$, $w$ near $p$? Justify your assertion(s).

c) If $c$ is one of these “good” values, let $G : (u, v, w) \mapsto (x, y, z)$ be the map inverse to $F$. Compute the derivative $G'(q)$ and use it to compute $\frac{\partial y(u, v, w)}{\partial v}$ at $q$. 


B–2. In a Hilbert space $\mathcal{H}$, let $v_1, \ldots, v_n$ be orthonormal vectors and $x \in \mathcal{H}$ a given vector.

a) Show there are scalars $a_1, \ldots, a_n$ and a $w \in \mathcal{H}$ with $w \perp \{v_1, \ldots, v_n\}$ so that

$$x = a_1v_1 + a_2v_2 + \cdots + a_nv_n + w.$$ 

Your work should exhibit a formula for the $a_k$ in terms of $x$ and $v_1, \ldots, v_n$.

b) Show that $||x||^2 = |a_1|^2 + \cdots + |a_n|^2 + ||w||^2$.

B–3. Let $u$ and $v$ be harmonic functions in a bounded (connected) region $D$ with $u = f$ and $v = g$ on the boundary of $D$. If $f < g$, show that $u < v$.

B–4. In homework you found that the Fourier series for $f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq \pi, \\ -1 & \text{for } -\pi \leq x < 0. \end{cases}$

is

$$f(x) \sim \frac{2}{i\pi} \left[ \left( \frac{e^{ix}}{1} + \frac{e^{3ix}}{3} + \frac{e^{5ix}}{5} + \cdots \right) - \left( \frac{e^{-ix}}{1} + \frac{e^{-3ix}}{3} + \frac{e^{-5ix}}{5} + \cdots \right) \right].$$

Use this and the Parseval Theorem to compute

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

B–5. Let $\varphi_k(x), x \in \mathbb{R}$, be a sequence of smooth functions with the following properties

i). $\varphi_k(x) \geq 0$ for $|x| < 1/k$, $\varphi_k(x) = 0$ for $|x| \geq 1/k$,

ii). $\int_{\mathbb{R}} \varphi_k(x) \, dx = 1$.

For a continuous function $f(x)$ with $f(x) = 0$ for $x$ outside a compact set $\mathcal{K}$, define

$$f_k(x) := \int_{\mathbb{R}} f(y)\varphi_k(x-y) \, dy.$$ 

Show that $\lim_{n \to \infty} f_k(x) = f(x)$, and that this convergence is uniform.