1. For which value(s) of the constant $c$ can the system of equations:

\[
\begin{align*}
 u(x,y,z) &= x + xyz \\
 v(x,y,z) &= y + xy \\
 w(x,y,z) &= z + cx + 3z^2
\end{align*}
\]

can be solved for $x$, $y$, $z$ as smooth functions of $u$, $v$, $w$ near $(1,1,0)$? Justify your assertion(s).

2. Let $y = f(x,u)$ and $z = g(x,u,v)$ be smooth functions with, say, $f(x_0,u_0) = y_0$ and $g(x_0,u_0,v_0) = z_0$.

a) Under what condition(s) can one eliminate $x$ from these equations to express $z$ as

\[z = F(y,u,v)\]

as a smooth function of $y$, $u$, and $v$ near $x = x_0$, $y = y_0$, $v = v_0$?

b) Assuming this, then compute $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial y}$ in terms of the derivatives of $f$ and $g$. To make this computation more specific, assume that

\[
\begin{align*}
 f_x(x_0,u_0) &= 1, & f_u(x_0,u_0) &= -2, & g_x(x_0,u_0,v_0) &= -3, & g_u(x_0,u_0,v_0) &= 4 \\
 & & g_v(x_0,u_0,v_0) &= -2.
\end{align*}
\]

3. Let $f(x)$, $a \leq x \leq b$ be a smooth function.

a) If $f(c) = 0$ for some $a \leq c \leq b$, show that

\[
|f(x)| \leq \int_a^b |f'(t)| \, dt \leq \sqrt{b-a} \left[ \int_a^b |f'(t)|^2 \, dt \right]^{1/2}.
\]

and hence, using the uniform norm $\|f\|_{\text{unif}} := \max_{a \leq x \leq b} |f(x)|$,

\[
\|f\|_{\text{unif}} \leq \int_a^b |f'(t)| \, dt \leq \sqrt{b-a} \left[ \int_a^b |f'(t)|^2 \, dt \right]^{1/2}.
\]

b) If $\int_a^b f(t) \, dt = 0$ (this replaces the assumption $f(c) = 0$), show that the above inequality still holds.

c) Use the result of part b) to show that for any smooth $f$

\[
\|f\|_{\text{unif}} \leq \int_a^b \left[ |f'(t)| + \frac{1}{b-a} |f(t)| \right] \, dt \leq c \left[ \int_a^b \left( |f'(t)|^2 + |f(t)|^2 \right) \, dt \right]^{1/2},
\]

where $c$ is a constant depending on $b-a$. [Suggestion: Apply the previous part to \(g := \hat{f} - \bar{f}\) where $\bar{f}$ is the average of $f$ over the interval.]
4. Say one has a real normed linear space whose norm comes from an inner product, so \( \|x\|^2 = \langle x, x \rangle \). Show that the norm satisfies the \textit{parallelogram identity}:
\[
\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.
\]
and interpret this identity geometrically.

5. [Review from Math 241] Let \( P_3 \) denote the space of real polynomials of degree at most three with the inner product
\[
\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx.
\]
Find an orthonormal basis for this space.

6. [Review from Math 241] Consider the space of complex-valued continuous functions \( C([-\pi, \pi]) \) with the inner product
\[
\langle f, g \rangle = \int_{-1}^{1} f(x)\overline{g(x)} \, dx,
\]
and let \( T_N \) be the linear space spanned by \( e^{ikx} \) for \(|k| \leq N\). Find the coefficients \( c_k \) so that
\[
x^2 = \sum_{|k| \leq N} c_k e^{ikx} + h(x),
\]
where \( h(x) \) is orthogonal to \( T_N \).

7. a) Let \( f(x) \in C([0,1]) \). If \( \int_{0}^{1} f(x)x^n \, dx = 0 \) for all \( n = 0, 1, 2, \ldots \), show that \( f \) must be identically zero.

b) Let \( f(x) \in C([0,1]) \). If \( \int_{0}^{1} f(x)x^{2k} \, dx = 0 \) for all \( k = 0, 1, 2, \ldots \), what can you conclude?

c) Let \( f(x) \in C([-1,1]) \). If \( \int_{-1}^{1} f(x)x^n \, dx = 0 \) for all \( n = 0, 1, 2, \ldots \), what can you conclude?

d) Let \( f(x) \in C([-1,1]) \). If \( \int_{-1}^{1} f(x)x^{2k} \, dx = 0 \) for all \( k = 0, 1, 2, \ldots \), what can you conclude?

8. a) Compute \( \min_{a,b,c} \int_{-1}^{1} |x^3 - ax - bx - cx^2|^2 \, dx \).

b) Compute \( \max \int_{-1}^{1} x^3 h(x) \, dx \) where \( h \in L^2(-1,1) \) is subject to the restrictions
\[
\int_{-1}^{1} h(x) \, dx = \int_{-1}^{1} xh(x) \, dx = \int_{-1}^{1} x^2 h(x) \, dx = 0; \quad \int_{-1}^{1} |h(x)|^2 \, dx = 1.
\]
9. [Dual variational problems] Let $V$ be a finite dimensional subspace of a Hilbert Space $H$ and $W$ its orthogonal complement. Recall that we can decompose any $x \in H$ uniquely as

$$x = x_V + x_W,$$

where $x_V \in V$, and $x_W \in W$.

a) Show that $\max_{\{z \in V, \|z\|=1\}} \langle x, z \rangle = \|x_W\|$.

b) Show that $\min_{v \in V} \|x - v\| = \|x_W\|$.

[Remark: dual variational problems are a pair of maximum and minimum problems whose extremal values are equal.]

**Bonus Problem 7.** Say one has a real normed linear space whose norm satisfies the parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Define the bilinear form $B(x, y)$ by the rule (see Problem 4 above)

$$B(x, y) := \frac{1}{4} \left[ \|x + y\|^2 - \|x - y\|^2 \right].$$

a) Show that $B(x, y)$ has all of the properties of an inner product including $B(x, x) = \|x\|^2$. [First observed by von Neumann]

**MORAL:** If a norm satisfies the parallelogram identity, then one can use it to define a compatible inner product; the parallelogram identity is both necessary and sufficient for the norm to arise from an inner product.

b) Consider $\mathbb{R}^2$ with vectors $X = (x_1, x_2)$ having the $p$ norm, $\|x\|_p := (|x_1|^p + |x_2|^p)^{1/p}$, $p \geq 1$.

Use the result of the previous problem to show that if $p \neq 2$, this norm does not arise from an inner product.

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