*Signature*                    PRINTED NAME

Math 609                                        Jerry L. Kazdan
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## Complex Analysis Exam I

DIRECTIONS  This exam has two parts, Part A has 7 short answer problems (35 points) while Part B has 5 traditional problems (65 points).

*Closed book* but you may use one $3 \times 5$ card with notes (on both sides).

All contour integrals are assumed to be in the positive sense (counterclockwise).

**Short Answer Problems** [5 points each] (35 points total)
For A1–A5 let $f(z)$ be holomorphic for $0 < |z| < \infty$. What can you say about $f(z)$ if you are told the following? Briefly justify your assertions.

A1. $|z^2 f(z)| < 5$.

| Score | |
|---|---|
| A1 | |
| A2 | |
| A3 | |
| A4 | |
| A5 | |
| A6 | |
| A7 | |
| B1 | |
| B2 | |
| B3 | |
| B4 | |
| B5 | |
| Total | |

A2. $|f(z)| \to \infty$ as $|z| \to 0$.

A3. $f''(z) + f(z) = 0$ for all real rational $z \neq 0$.

A4. $|f(z)| \leq |z| + 1$ and $f(\frac{1}{n}) = 0$, $\quad n = 1, 2, \ldots$ .

A5. $|f(z)| \leq |f(3)|$ for $|z - 3| < 2$.

A6. If $\sum_{n=0}^{\infty} a_n z^n$ represents the function $\dfrac{\sin z}{z^2 + 2}$, whch of the following are true? (Why?)

    (A). Converges for $z = 1$.

    (B). Converges absolutely for $z = 1$.

    (C). Converges absolutely for $z = 2$.

A7. Describe the singularities of $\varphi(z) := \dfrac{1 - \cos(z^5)}{\sin^3 z}$ at $z = 0$, $z = \pi$, and $z = \infty$.

**Traditional Problems** [13 points each] (65 points total)

B1. Prove the Fundamental Theorem of Algebra: that any polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0, \quad n > 0,$$

with complex coefficients $a_k$ has exactly $n$ complex roots, counted with multiplicity.

B2. Let $f(z)$ be an entire function that is a one-to-one map from $\mathbb{C}$ to $\mathbb{C}$. Show that $f(z) = az + b$ for some complex constants $a \neq 0$ and $b$.

B3. Evaluate $A = \displaystyle\int_0^\infty \frac{\cos xt}{x^2 + a^2}\, dx$  where $a > 0$ and $t > 0$.

B4. Let $h(z)$, $z = x + iy$, be holomorphic in the strip $|y| < 10$ with $|h(z)| < 1$ there. Prove that $\cos z + h(z)$ has an infinite number of zeroes in this strip. [NOTE: You may use without proof that $|\cos z|^2 = \cosh^2 y - \sin^2 x$].

B5. Let a function $f(z)$ have all of the properties

(a). holomorphic in $\{|z| \le 1\}$,

(b). $|f(z)| = 1$ for $|z| = 1$,

(c). $f(a) = 0$ for some $|a| < 1$,

(d). $f(z) \ne 0$ for all $|z| \le 1$, with $z \ne a$.

Prove that $f(z) = \alpha \left( \dfrac{z-a}{1-\bar{a}z} \right)^k$, where $|\alpha| = 1$ and $k$ is a positive integer.