Final Examination

Directions: Answer all 5 questions. Time: One hour. You may use one sheet of A4 paper with notes on one side. Try to communicate your ideas clearly.

1. Let \( f \in C^2(\mathbb{R}) \) be a \( 2\pi \) periodic function, so \( f \) and its first two derivatives are \( 2\pi \) periodic. Say \( f(x) = \sum_k c_k e^{ikx} \) is its Fourier series.
   a) Show there is a constant \( m \) so that \( |c_k| \leq \frac{m}{1+k^2} \).
   b) Show that the Fourier series converges uniformly.

2. Let \( u(x,t) \) be a solution of \( u_{tt} + b(x,t)u_t = u_{xx} \) for \( 0 < x < L \). Assume \( u \) satisfies the initial conditions \( u(x,0) = 0 \) and \( u_t(x,0) = 0 \) and boundary conditions \( u(0,t) = u(L,t) = 0 \).
   a) If \( b(x,t) \geq 0 \), show that \( u(x,t) = 0 \) for all \( t > 0 \).
   b) If \( |b(x,t)| \leq M \) for some constant \( M \) show that \( u(x,t) = 0 \) for all \( t > 0 \).

3. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open set and \( u(x,t) \) a solution of \( u_t = \Delta u \) in \( \Omega \) with \( u(x,t) = f(x) \) for \( x \in \partial \Omega \). Also, let \( v(x) \) satisfy \( \Delta v = 0 \) in \( \Omega \) with \( v(x) = f(x) \) on \( \partial \Omega \). Show that, in an appropriate sense, \( \lim_{t \to \infty} u(x,t) = v(x) \).

4. Let \( \Omega \subset \mathbb{R}^3 \) be a bounded open set. Assume \( Lu := -\Delta u + c(x)u \geq 0 \), where \( c(x) > 0 \) is a continuous function.
   a) Show that \( u \) cannot assume a negative minimum at any point of \( \Omega \).
   b) If \( u \) and \( v \) satisfy \( Lu = f \) and \( Lv = g \), respectively, in \( \Omega \) with \( f > g \) in \( \Omega \) and \( u = v \) on \( \partial \Omega \), what can you conclude? Proof?

5. Pick a topic (or technique) in the course that interested you and give a brief summary of it. You may include theorems, proofs, ideas, examples, special cases, etc. You don’t need to be really precise, but give the main ideas – as if you were describing it to a friend at coffee.
   [Please don’t jabber. First think and plan calmly. Please do not write more than one page.]