MATH 240 Quiz 11 Name: ____________________

Question:

Determine the general solution to the system $\mathbf{x}(t)' = A\mathbf{x}(t)$ where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Extra credit(+2): Suppose a solution $\mathbf{x}(t)$ to the above equation satisfies

$$\lim_{t \to +\infty} \mathbf{x}(t) = \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}.$$

where $-1 < a, b < 1$, determine a and b.

Solution:
Notice in this case, A is diagonal, so its exponential is convenient to compute, and general solution is:

$$e^{At}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 e^t \\ c_3 e^{-t} \end{pmatrix}.$$

If $c_2 \neq 0$, $ce^t \to \infty$ as $t \to +\infty$, then a will not be finite. So we need $c_2 = 0$, therefore $a = 0$. No matter what c_3 is, $c_3 e^{-t} \to 0$ as $t \to +\infty$, so $b = 0$.
Another appoach is: the limit is a solution (steady state solution) to the system of differential equation, plug in, we get:

$$\frac{d}{dt}\begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ b \end{pmatrix}$$

therefore $a = b = 0$.