Hopf Subalgebras with
Algebraic Quotient Modules

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Consider a Hopf subalgebra pair $R \subseteq H$, where $H$ is a finite-dimensional Hopf algebra over an arbitrary field.

"Quotient" $V = H/R^+H$ where $R^+ = \ker \varepsilon_R$, a right $H$-module coalgebra via

$$\Delta_V(h) = h(1) \otimes h(2).$$

Finite tensor category $U, W \in \mathcal{M}_H$:

$$(u \otimes w) \cdot h = u \cdot h(1) \otimes w \cdot h(2)$$

Notation: $H \otimes^R n := H \otimes_R \cdots \otimes_R H$ ($n$ times $H$)

Basic Lemma. $H \otimes^R n \cong H \otimes V \otimes^{(n-1)}$ via

$$x \otimes y \otimes \cdots \otimes z \mapsto xy(1) \cdots z(1) \otimes y(2) \cdots z(2) \otimes \cdots \otimes z(n),$$

with inverse mapping given by

$$u \otimes v \otimes w \otimes \cdots \mapsto uS(v(1)) \otimes_R v(2)S(w(1)) \otimes_R w(2) \cdots.$$
Similarity of two modules, $X \sim Y$ if $X \oplus \ast \cong Y \oplus \cdots \oplus Y$, briefly $X \mid n \cdot Y$ and $Y \mid m \cdot X$ for some $m, n \in \mathbb{N}$.

If $X, Y \in$ Krull-Schmidt category, and indecomposable isoclass summands of $X$ denoted by $\text{Indec}(X)$, then

$X \mid n \cdot Y \Leftrightarrow \text{Indec}(X) \subseteq \text{Indec}(Y)$ and $X \sim Y \Leftrightarrow \text{Indec}(X) = \text{Indec}(Y)$

A subalgebra $R \subseteq H$ has finite depth $(2n + 1)$ if for some $n \in \mathbb{N}$, $H \otimes_R^n \sim H \otimes_R^{(n+1)}$ as $X,Y$-bimodules for any $X,Y \in \{R,H\}$ ($X = Y = R$).

Lemma. If any of $R^e$, $H^e$, $R \otimes H^{\text{op}}$ or $H \otimes R^{\text{op}}$ has finite representation type, then $R \subseteq H$ has finite depth.

Proof follows from $H \otimes_R^m \mid H \otimes_R^{(m+1)}$. 
Def. A module algebra or coalgebra $W$ in $\mathcal{M}_H$ has *finite depth* (depth $n$) if $W \otimes n \sim W \otimes (n+1)$ in $\mathcal{M}_H$ for some $n \in \mathbb{N}$.

Let $A(H)$ be the Green ring (under $\oplus$ and $\otimes$ w.r.t. isoclasses, basis of indecomposables).

Theorem. A Hopf subalgebra $R \subseteq H$ has finite depth iff its quotient $V$ is algebraic in $A(H)$ or in $A(R)$.

PF. W. Feit’s text: an $H$-module is algebraic if it satisfies a polynomial equation in $A(H)$. This is equivalent to finite depth module and carries over to Hopf algebras from group alg’s. $(\Leftarrow)$ If $V \otimes m \sim V \otimes (m+1)$ in $\mathcal{M}_R$, then $H \otimes_R (m+1) \cong H \cdot H \otimes V \cdot m \sim H \cdot H \otimes V \cdot (m+1) \cong H \otimes_R (m+2)$ as $H$-$R$-bimodules.

$(\Rightarrow)$ Permute the argument above and apply $k \otimes_H -$.

Corollary. Depth is finite in case $V$ is in finite rank ideal of either Green ring.
Example: $R_d = \text{Taft algebra} \subseteq H_d = \overline{U}_q(sl_2(\mathbb{C}))$ (small quantum group) where $q = \text{prim. } n'^{th}$ root of unity. ($d = n$ if $n$ odd, $d = \frac{n}{2}$ if $n$ even.) $H_d$ gen. by $K, E, F$ with relations $K^d = 1$, $E^d = 0 = F^d$, $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$, $KE = q^2EK$, and $KF = q^{-2}FK$.

coalgebra: $\Delta(K) = K \otimes K$, $\Delta(E) = E \otimes 1 + K \otimes E$ and $\Delta(F) = F \otimes K^{-1} + 1 \otimes F$. $\varepsilon(K) = 1$, $\varepsilon(E) = 0 = \varepsilon(F)$.

Theorem (Li-Liu, 2006). A basic Hopf $\mathbb{C}$-algebra has finite rep. type if and only if it is a Nakayama algebra: each projective indecomposable has unique composition series $\iff$ each vertex of quiver has at most 1 incoming and 1 outgoing arrow.
The ordinary quiver of $H_d$: the orthog. prim. idempotent $e_{i+1} = \sum_{j=0}^{d-1} (q^{2i} K)^j / d$ where

\[ e_{i+1} E = E e_{i+2} \in e_{i+1} (J/J^2) e_{i+2}, \]

\[ F e_{i+1} = e_{i+2} F \in e_{i+2} (J/J^2) e_{i+1}. \] Not Nakayama!

For $d = 4$:

\[
\bullet^1 \iff \bullet^2 \\
\uparrow \downarrow \quad \uparrow \downarrow \\
\bullet^4 \iff \bullet^3
\]

Hopf subalgebra $= \text{Taft algebra } R_d$ gen. by just $K, F$ with relations $K^d = 1, F^d = 0, KF = q^{-2}FK$. Same set of orth. prim. idemp. so quiver has 1 outgoing, 1 incoming arrow at each vertex: $R_d$ Nakayama! E.g., at $d = 4$:

\[
\bullet^1 \leftrightarrow \bullet^2 \\
\downarrow \quad \uparrow \\
\bullet^4 \rightarrow \bullet^3
\]

Since $R_d$ has fin. rep. type, the Hopf subalgebra $R_d \subset H_d$ has finite depth.
$V = H_d/R_d^+H_d$ spanned by $\{\overline{1}, \overline{E}, \ldots, \overline{E}^{d-1}\}$. Semisimple for $d = 2$, not semisimple when $d \geq 3$ since $VJ \neq 0$ from $\overline{E^2} \cdot F = -(q + q^{-1})\overline{E}$.

Theorem. A Hopf subalgebra $R \subseteq H$ with semisimple quotient $V_R$ has finite depth if $R$ has the Chevalley property.

PF. Chevalley property: tensor product of semisimple modules remains semisimple. Then $V \otimes^n$ remains a direct sum of finitely many simples for all $n \geq 0$. Apply previous theorem.

Example. The duals of pointed Hopf algebras, such as the basic algebras $R_d$ and $H_d$ above, have this property.
Proposition. A Hopf subalgebra $R \subseteq H$ with projective $V_R$ has finite depth. However, $V_R$ projective $\iff$ $R$ semisimple!

PF. $0 \rightarrow R^+ \rightarrow R \xrightarrow{\xi} k \rightarrow 0$ by definition. Apply free (and faithfully flat) functor of induction from $R$ up to $H$:

$$0 \rightarrow R^+ H \rightarrow H \rightarrow V \rightarrow 0$$

Finally the projectives of $A(R)$ form a finite rank ideal.

Group Example. Let $H = k[G]$ and $R = k[K]$ where $K \subseteq G$ is a finite group-subgroup pair. Then $V = \mathbb{C}[G/K]$ the permutation module of right cosets (via $\bar{g} \mapsto Kg$).

Feit, Chapter 9: Permutation modules are algebraic modules. Consequently, $H \supseteq R$ has finite depth.
Group Example Continued. \( k = \mathbb{C} \). The character of \( V \) is the induced character \( 1^G_K \), which is \textit{faithful} if \( K \) is a \textit{corefree} subgroup in \( G \).

Brauer-Burnside Theorem. The powers of a faithful character \( \chi \) contain each irreducible character of \( G \)....
\[ \Rightarrow \text{minimum depth of } V \text{ less than no. of distinct values assumed by } 1^G_K \text{ on } G. \]

\( k \) a general field again. \( R^+ \supset \text{Ann}_R V \supset \cdots \supset \text{Ann}_R V \otimes \ell_V = \text{Ann}_R V \otimes (\ell_V + 1) = \cdots = \text{maximal Hopf ideal in } \text{Ann}_R V \).

Theorem. If \( R \) is semisimple with \( n \) irreducible characters, and \( V \) has annihilator ideal that contains no nonzero Hopf ideal (or bi-ideal), then minimum depth of \( R \subseteq H \) is less than \( 2\ell_V + 2 \), where \( \ell_V \leq n \) is the least tensor power of \( V \) that is faithful \( R \)-module.
\[ V = H/R^+H \] a right \( H \)-module coalgebra ⇔ \( V^* \) a left \( H \)-module algebra.

Theorem. If \( A \) is a left \( H \)-module algebra, then

\[
(A \# H)^{\otimes H^n} \cong .A^{\otimes n} \otimes .H
\]

as \( H-H \)-bimodules.

Corollary 1. \( H \) has finite depth in \( V^* \# H \) ⇔ \( V^* \) has finite depth ⇔ \( V \) has finite depth ⇔ Hopf subalgebra \( R \) has finite depth in \( H \).

Corollary 2. Let \( G \) be finite centerless group. Then depth \( d(\mathbb{C} G, D(G)) = 2\ell_V + 1 \), where \( V \) is the adjoint representation of \( G \) on itself.

Corollary 3. Let \( \dim H \geq 2 \). Then the minimum odd depth of \( H^* \) in its Heisenberg double satisfies \( d_{odd}(H^*, H \# H^*) = 3 \).
Spinoff and Further Thoughts

Theorem. $R$ is ad-stable in $H$ if and only if right integral $t \in R$ is normal element in $H$.

PF. ($\Leftarrow$) Normal element $t \in R \subseteq H$ if $tH = Ht$.

But $V \overset{\sim}{\rightarrow} tH$ via $h + R^+ H \mapsto th$. Then $V_R \cong Ht$ has depth 0, so depth of $R$ in $H$ is 2. This is equivalent to $R$ being ad-stable in $H$ by Boltje-Külshammer characterization.

- Algebra $V^*$ is Frobenius.

- Some generalization to $K$ a left coideal subalgebra of $H$ : subalgebra depth is finite iff $V_H$ is algebraic module.
• The dual monic $V^* \hookrightarrow H^*$ is left ($R^*$-) Hopf-Galois extension, with normal basis property: $H^* \cong V^* \#_\sigma R^*$.

• Are generalized permutation modules $V \cong k \otimes_R H$ algebraic? Not clear without Mackey theorems.
Thanks for tuning in!

References


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D. Craven, Simple modules for groups with abelian Sylow 2-subgroups are algebraic, *J. Algebra* 321 (2009), 1473–1479.
Earlier articles by Alperin, Berger.