Subring depth below an ideal

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In memory of Gerhard Hochschild

Abstract. A minimum depth is assigned to a ring homomorphism and a bimodule over its codomain. When the homomorphism is an inclusion and the bimodule is the codomain, the recent notion of depth of a subring in a paper by Boltje-Danz-Külshammer is recovered. Subring depth below an ideal gives a lower bound for BDK’s subring depth of a group algebra pair or a semisimple complex algebra pair.

1. Introduction and Preliminaries

Algebras, coalgebras and Hopf algebras are some of the interesting objects with structure in representation categories of commutative rings. In the representation category of a noncommutative ring, these objects become ring extensions, corings and Hopf algebroids. Some basic algebras of interest are the cohomological dimension 0 and ∞ cases of separable algebra and Frobenius algebra; which become separable extensions and Frobenius extensions in noncommutative representation theory. Also, QF rings, semisimple rings, and Azumaya algebras generalize to ring extensions; however depth is not such a notion, originating as a tool of induced representation theory. Depth is essentially constant on (especially projective) algebras over a commutative ring, but gives different and interesting outcomes for ring extensions.

The depth of many subgroups are recently computed, both as induced complex representations [6] and as induced representations over general commutative rings of group algebras [1]. For example, the depth of the permutation groups $S_n \subset S_{n+1}$ is $2n - 1$ over any ground ring and depends only on a combinatorial depth of subgroups defined in terms of bisets in [1]. The authors of [1] show that combinatorial depth $d_c(H,G)$ of a subgroup $H$ in a finite group $G$ satisfies $d_c(H,G) \leq 2n$ for $n \geq 1$ (respectively, $d_c(H,G) \leq 2n - 1$ for $n > 1$) if for any $x_1, \ldots, x_n \in G$, there is $y_1, \ldots, y_{n-1} \in G$ such that $H \cap_{i=1}^{n} x_i H x_i^{-1} = H \cap_{i=1}^{n-1} y_i H y_i^{-1}$ (respectively, the latter condition and additionally $x_1 h x_1^{-1} = y_1 h y_1^{-1}$, all $h \in H \cap_{i=1}^{n} x_i H x_i^{-1}$). All notions of depth $\leq 2$ are the same and occur precisely if $H$ is a normal subgroup. However, depth of subalgebras other base rings (of varying characteristic denoted by a subscript) for $R = k[G]$ and $S = k[H]$ and combinatorial depth diverge in a string of inequalities given in [1] as follows:

$$d_0(H,G) \leq d_p(H,G) \leq d_Z(H,G) \leq d_c(H,G) \leq 2[G : N_G(H)].$$  \hfill (1)
Also \( d_k(H,G) \leq d_e(H,G) \) showing that all extensions of finite dimensional group algebras have finite depth.

The authors begin in [1] with a new notion of subring depth \( d(S, R) \), given below in (4). They show in an appendix how it is based on and equal to a previous notion where \( S \) and \( R \) are semisimple complex algebras given below in (5). Such a pair \( R \supseteq S \) is a special case of (split separable) Frobenius extensions; in [13, Theorem 5.3] I show that subring depth is equal to tower depth of Frobenius extensions [11] satisfying a generator module condition. The authors of [1] define a left and right even depth and show these are the same on group algebra extensions; [13, Theorem 3.4] shows this equality holds for all QF extensions.

In this paper a change is made to the definition of subring depth; we define an I-depth \( d^I(S \to R) \) of a ring homomorphism \( S \to R \) with \( R \)-bimodule \( I \), which we use in place of \( R \) in the n-fold tensor products over \( S \) in the definition (4) of \( d(S, R) \) (as well as a converse, automatic in the presence of units). When \( I \) is an ideal of a semisimple complex algebra \( R \) with semisimple subalgebra \( S \) the I-depth \( d^I(S \to R) \) gives a lower bound, \( d^I(S \to R) \leq d(S, R) \) discussed in Section 2 in terms of the part of the bipartite graph of the inclusion which is directly below the ideal \( I \).

There are intriguing similarities and relations between relative homological algebra and subring depth theory. For example, the depth two condition on a subring \( S \subseteq R \) leads in [10] to an isomorphism of differential graded algebras between the relative Hochschild \( R \)-valued cochains with cup product and the Amitsur complex of a coring with grouplike element (on the endomorphism ring \( \text{End} \, g \mathcal{R} \mathcal{S} \) over the centralizer subring \( R^S \)). Also the paper [12] contains some relations between depth 2 and notions of relative homological algebra carried over to corings in [4]. The tower of iterated endomorphism rings above a ring extension becomes in the case of Frobenius extensions a tower of rings on the bar resolution groups \( C_n(R, S) \) \((n = 0, 1, 2, \ldots)\) with Frobenius and Temperley-Lieb structures explicitly calculated from their more usual iterative definition in the paper [13]. At the same time Frobenius extensions of depth more than 2 are known to have depth 2 further out in the tower: I extend this observation in [11] with new proofs to include other ring extensions satisfying the hypotheses of [13, Prop. 4.3].

### 1.1. H-equivalent modules

Let \( R \) be a ring. Two left \( R \)-modules, \( R N \) and \( R M \), are said to be h-equivalent, denoted by \( R M \not\sim R N \) if two conditions are met. First, for some positive integer \( r \), \( N \) is isomorphic to a direct summand in the direct sum of \( r \) copies of \( M \), denoted by

\[
R N \oplus \ast \cong R M^r \quad \Leftrightarrow \quad N \mid M^r \Leftrightarrow
\]

\[
\exists f_i \in \text{Hom} \left( R M, R N \right), \ g_i \in \text{Hom} \left( R N, R M \right), i = 1, \ldots, r : \sum_{i=1}^{r} f_i \circ g_i = \text{id}_{N}
\]

Second, symmetrically there is \( s \in \mathbb{Z}_+ \) such that \( M \mid N^s \). It is easy to extend this definition of h-equivalence (sometimes referred to as similarity) to h-equivalence of two objects in an abelian category, and to show that it is an equivalence relation.

If two modules are h-equivalent, \( R N \not\sim R M \), then they have Morita equivalent endomorphism rings, \( \mathcal{E}_N := \text{End} \, R N \) and \( \mathcal{E}_M := \text{End} \, R M \). This is quite easy to see since a Morita context of bimodules is given by \( H(M,N) := \text{Hom} \left( R M, R N \right) \), which is an \( \mathcal{E}_N \)-\( \mathcal{E}_M \)-bimodule via composition, and the bimodule \( \mathcal{E}_M H(N,M) \mathcal{E}_N \); these are progenerator modules, by applying to (2) or its reverse, \( M \mid N^s \), any of the four Hom-functors such as \( \text{Hom} \left( R \_, R M \right) \) from the category of left \( R \)-modules into the category of left \( E_M \)-modules showing that \( \mathcal{E}_M H(N,M) \) is finite projective; similarly, generator. Then the explicit conditions on mappings for h-equivalence show that \( H(M,N) \otimes_{\mathcal{E}_M} H(N,M) \to \mathcal{E}_N \) and the reverse mapping given by composition are
both bimodule isomorphisms as required. Since \( \mathcal{E}_M \) and \( \mathcal{E}_N \) are Morita equivalent rings, their centers are isomorphic:

\[
\text{End}_R \mathcal{E}_M \cong \text{End}_R \mathcal{E}_N.
\]

The theory of h-equivalent modules applies to bimodules \( \tau M_S \cong \tau N_S \) by letting \( R = T \otimes \mathbb{Z} \otimes S^{op} \) which sets up an equivalence of abelian categories between \( T\)-\( S \)-bimodules and left \( R \)-modules. Two additive functors \( F, G : C \rightarrow \mathcal{D} \) are h-equivalent if there are natural split epis \( F(X)^n \rightarrow G(X) \) and \( G(X)^m \rightarrow F(X) \) for all \( X \) in \( C \). We leave the proof of the lemma below as an elementary exercise.

**Lemma 1.1** Suppose two \( R \)-modules are h-equivalent, \( M \cong N \) and two additive functors from \( R \)-modules to an abelian category are h-equivalent, \( F \cong G \). Then \( F(M) \cong G(N) \).

For example, the following substitution in equations involving the \( \cong \) -equivalence relation follows from the lemma:

\[
RPT \cong RQT \quad \tau US \cong \tau VS \quad \Rightarrow \quad RP \otimes \tau US \cong RQ \otimes \tau VS
\]

**Example 1.2** If \( R \) is a semisimple artin ring with simples \( \{ P_1, \ldots, P_t \} \) (representatives from each isomorphism class), all finitely generated modules \( M_R \) and \( N_R \) have a unique factorization into simple components. Denote the simple constituents of \( M_R \) by \( \text{Simples}(M) = \{ P_i \mid [P_i, M] \neq 0 \} \) where \( [P_i, M] \) is the number of factors in \( M \) isomorphic to \( P_i \). Then \( M \mid N^q \) for some positive \( q \) if \( \text{Simples}(M) \subseteq \text{Simples}(N) \); and \( M \cong N \) iff \( \text{Simples}(M) = \text{Simples}(N) \).

Suppose \( R \) has central primitive idempotents \( e_1, \ldots, e_t \) such that each \( [P_i, e_i R] = n_i \), so that \( R \) decomposes into the product of matrix rings over each of the division rings \( D_i := \text{End}(P_i)_R \): \( R \cong \bigoplus_{i=1}^{n_t} (D_1) \times \cdots \times (D_t) \). If \( M \) and \( N \) are h-equivalent f.g. \( R \)-modules, then the endomorphism rings \( E_M \) and \( E_N \) are explicitly Morita equivalent as they are both products of matrix rings over the same subset of division rings \( D_1, \ldots, D_t \).

**Example 1.3** Via some more category theory, we may see that positive integers \( n \) and \( m \) are h-equivalent if \( n \mid m' \) and \( m \mid n' \) for some positive integers \( r, s \); whence there are primes \( p_1, \ldots, p_k \) such that \( n \) and \( m \) lie in the same h-equivalence class \( \{ p_1^{r_1} \cdots p_k^{r_k} \mid r_1, \ldots, r_k \geq 1 \} \). This explains the notation in (2).

### 1.2. Depth two

A subring pair \( S \subseteq R \) is said to have left depth 2 (or be a left depth two extension [9]) if \( R \otimes S R \cong R \) as natural \( S \)-\( R \)-bimodules. Right depth 2 is defined similarly in terms of h-equivalence of natural \( R \)-\( S \)-bimodules. In [9] it was noted that the left condition implies the right and conversely if \( R \) is a Frobenius extension of \( S \). Also in [9] a Galois theory of Hopf algebroids was defined on the endomorphism ring \( H := \text{End}_S R_S \) as total ring and the centralizer \( C := R^S \) as base ring. The antipode is the restriction of the natural anti-isomorphism stemming from following the arrows,

\[
\text{End}_R S \xrightarrow{\sim} R \otimes S R \xrightarrow{\sim} (\text{End}_S R)^{op}.
\]

The Galois properties may then be summarized by the invariants under the obvious action of \( H \), \( R^H = S \) if \( R_S \) is faithfully flat, and \( \text{End}_R S \cong R^H H \) a smash product product ring structure on \( R \otimes C \) \( H \): the details are in [9]. There is also a duality structure by going a step further along in the tower above \( S \subseteq R \rightarrow \text{End}_R S \rightarrow \text{End}_R \otimes_S R_R \) where the dual Hopf algebroid \( H' := (R \otimes_S R)^{op} \) plays a role [9].

Conversely, Galois extensions have depth 2, which is most easily seen from the Galois map of an \( H \)-comodule algebra \( A \) with invariant subalgebra \( B \) and finite dimensional Hopf algebra
$H$ over a base field $k$, which is given by $A \otimes_B A \xrightarrow{\cong} A \otimes_k H$, $a' \otimes a \mapsto a'a_{(0)} \otimes a_{(1)}$, whence $A \otimes_B A \cong A^{\text{dim}H}$ as $A$-$B$-bimodules. The Hopf subalgebras within a finite dimensional Hopf algebra which have depth 2 are precisely the normal Hopf subalgebras; if normal, it has depth 2 by applying the Hopf-Galois observation just made. The converse follows from an argument discovered by [2, Boltje-Külshammer] which divides the normality notion into right and left just like depth 2, where left normal is invariance under the left adjoint action. Note their argument given in the context of any augmented algebra $A$ (such as a quasi-Hopf algebra) next. Let $\varepsilon : A \to k$ be the algebra homomorphism into a base ring $k$. Let $A^+$ denote $\ker \varepsilon$, and for a subalgebra $B \subseteq A$, let $B^+$ denote $\varepsilon \cap B$. For example, it may be shown that if a (quasi-)Hopf algebra $H$ has (quasi-)Hopf subalgebra that is invariant under the left adjoint action of $H$, then $HK^+ \subseteq K^+H$.

**Proposition 1.4** Suppose $B \subseteq A$ is a subalgebra of an augmented algebra. If $B \subseteq A$ has right depth 2, then $AB^+ \subseteq B^+A$.

**Proof.** To $A \otimes_B A | A^g$ as $A$-$B$-bimodules, apply the additive functor $k \otimes A$ $-\rightrightarrows$, which results in $A/B^+A | k^g$ as right $B$-modules. The annihilator of $k^g$ restricted to $B$ is of course $B^+$, which then also annihilates $A/B^+A$, so $AB^+ \subseteq B^+A$.

The opposite inclusion is of course satisfied by a left depth 2 extension of augmented algebras.

Also subalgebra pairs of semisimple complex algebras have depth 2 exactly when they are normal in a classical sense of Rieffel. We note the theorem in [6] below and give a new proof in one direction along the lines of the previous proposition.

**Theorem 1.5** [6, Theorem 4.6] Suppose $B \subseteq A$ is a subalgebra pair of semisimple complex algebras. Then $B \subseteq A$ has depth 2 if and only if for every maximal ideal $I$ in $A$, one has $A(I \cap B) = (I \cap B)A$.

**Proof.** ($\Leftarrow$) See [6, Section 4]. ($\Rightarrow$) Given maximal ideal $I$ in $A$, there is an ideal $J$ with identity element $1_J$ such that $A = I \oplus J$, and algebra homomorphism $\varepsilon : A \to A/I \cong J$. Denote $I = A^+$, $I \cap B = B^+$, and note that the $A$-module $J_A = J_\varepsilon$. Given the right depth 2 condition $A_A \otimes_B A_B | A^g$, tensor from the left by $J_A$ obtaining $J \otimes_B A_B | J^g$.

Note the $B$-module homomorphism $A/B^+A \to J \otimes B A$ given by $a + B^+A \mapsto 1_J \otimes B a$ (well-defined since $1_J \cdot B^+ = 0$) which we claim is monic. For suppose that $1_J \otimes a = 0$ for $a$ in the projective module $B A$, so $a = \sum_i f_i(a) e_i$ in some free module $B^n$. Then

$$0 = 1_J \otimes a = \sum_i \varepsilon(f_i(a)) \otimes e_i \Rightarrow \varepsilon(f_i(a)) = 0 \Rightarrow f_i(a) \in B^+, \forall i = 1, \ldots, n$$

hence $a = \sum_i f_i(a) e_i \in B^+ A$ so $a + B^+A = 0$ which proves the claim.

Since $B^+$ annihilates $J^g_B$, it annihilates $J \otimes_B A_B$ and therefore $A/B^+A$ via the monomorphism. Thus $AB^+ \subseteq B^+A$. The opposite inclusion follows from a similar argument applied to the left depth 2 condition.

2. Ideal depth of a ring homomorphism

Let $S$ and $R$ be unital associative rings and $S \to R$ a ring homomorphism where $1_S \mapsto 1_R$. Suppose $R/I_R$ is a bimodule. With no further ado, we will restrict $I$ to bimodules $S I_R$, $R I_S$ or $S I_S$ via the homomorphism $S \to R$. Note that the kernel of $S \to R$ is contained in the annihilator ideal in $S$ of the left (or right) $S$-module $I$ denoted by $\text{ann}_S I$.

We let $C^0_I(S \to R) = S$, and for $n \geq 1$,

$$C^n_I(S \to R) = I \otimes_S \cdots \otimes_S I \quad (n \text{ times} I)$$

For $n \geq 1$, the $C^n_I(S \to R)$ has a natural $R$-$R$-bimodule (briefly $R$-bimodule) structure which restricts to $S$-$R$-, $R$-$S$- and $S$-bimodule structures occurring in the next definition.
Definition 2.1 The ring homomorphism \( S \to R \) has \( I \)-depth \( 2n + 1 \geq 1 \) if as \( S \)-bimodules \( C_n^I(S \to R) \cong C_{n+1}^I(S \to R) \). The ring homomorphism \( S \to R \) has left (right) \( I \)-depth \( 2n \geq 2 \) if \( C_n^I(S \to R) \cong C_{n+1}^I(S \to R) \) as \( S \)-\( R \)-bimodules (respectively, \( R \)-\( S \)-bimodules).

It is clear that if \( S \to R \) has either \( I \)-depth \( 2n \), it has \( I \)-depth \( 2n + 1 \) by restricting the \( h \)-equivalence condition to \( S \)-bimodules. If it has \( I \)-depth \( 2n + 1 \), it has \( I \)-depth \( 2n + 2 \) by tensoring the \( h \)-equivalence by \(- \otimes_S I \) or \( I \otimes_S - \). The minimum \( I \)-depth is denoted by \( d^I(S \to R) \).

Note that the minimum left and right minimum even \( I \)-depths may differ by \( 2 \) (in which case \( d^I(S \to R) \) is the least of the two). In the next section we provide a general condition, which includes a Hopf subalgebra pair \( S \subseteq R \) of symmetric Frobenius algebras with \( I \) an ideal in \( R \), where the left and right minimum even \( I \)-depths coincide.

We also remark that once \( S \to R \) has \( I \)-depth \( 2n + 1 \) the \( C_n^I(S \to R) \)'s stop growing as \( m \to \infty \) in terms of adding new indecomposables in a category of modules with unique factorization, since \( C_n^I(S \to R) \cong C_{n+m}^I(S \to R) \) for all \( m \geq 0 \) (see the example in the previous section). This corresponds well with the classical notion of finite depth in subfactor theory.

Lemma 2.2 Let \( S \to R \) have kernel \( K \), \( \overline{S} := S/K \) and \( \overline{S} \to R \) be the induced ring homomorphism. Then the left or right minimum depth \( d^I(S \to R) = d^I(\overline{S} \to R) \) unless \( d^I(\overline{S} \to R) = 1 \), in which case equality holds if the quotient homomorphism \( p : S \to \overline{S} \) has a section.

Proof. Note that if \( M_R | N_R \), then \( \text{ann} N_R \subseteq \text{ann} M_R \). Since \( K \) is in \( \text{ann} C_n^I(R,S) \) for all \( n \geq 1 \) and \( C_n^I(R,S) \cong C_n^I(R,\overline{S}) \) as \( \overline{S} \)-modules, it follows that \( C_n^I(R,S) \cong C_n^I(R,\overline{S}) \) implies \( C_n^I(R,\overline{S}) \cong C_n^I(R,\overline{S}) \) for the bimodules at issue. The converse is easy by pullback along \( p \).

\( S \to R \) has \( I \)-depth \( 1 \) iff there are central elements \( w_i, z_i \in I^S \) and mappings \( f_j, g_i \in \text{Hom}(\sigma I_S, S) \) such that \( x = \sum_i z_i g_i(x) \) for all \( x \in I \) and \( \sum_j f_j(w_j) = 1_S \). By composing with the quotient homomorphism \( S \to \overline{S} \), we obtain \( \tilde{f}_j, \tilde{g}_i \in \text{Hom}(\overline{S}I_S, \overline{S}) \) and \( z_i \in I^\overline{S} \) such that \( x = \sum_i \tilde{g}_i(x) z_i \) and \( \sum_j \tilde{f}_j(w_j) = 1_{\overline{S}} \). The converse may be proven with the extra hypothesis in the lemma, since all mappings in \( \text{Hom}(\overline{S}I_S, \overline{S}) \) have a lifting to \( \text{Hom}(\sigma I_S, \overline{S}) \) along \( p \) via a section \( \sigma : \overline{S} \to S \) satisfying \( p \circ \sigma = \text{id}_{\overline{S}} \).

Example 2.3 Suppose \( S \) is a subring of \( R \) (where \( 1_S = 1_R \)). Let \( S \to R \) be the inclusion monomorphism and \( I = R \), the natural \( R \)-bimodule. The minimum depth of the subring \( S \subseteq R \) as defined in [1, Boltje-Danz-Külshammer] is denoted by \( d(S,R) \). We note that \( d(S,R) = d^R(S \to R) \). In fact, \( C_n^R(S \to R) = R \otimes_S \cdots \otimes_S R := C_n(R,S) \) (n times \( R \)) for \( n > 0 \), and the depth \( 2n + 1 \) condition in [1] is that

\[
C_{n+1}(R,S) \mid C_n(R,S)^q
\]

(4)

as \( S \)-bimodules (some \( q \in \mathbb{Z}_+ \)). The left depth \( 2n \) condition in [1] is (4) more strongly as natural \( S \)-\( R \)-bimodules (and as \( R \)-\( S \)-bimodules for the right depth 2n condition). But (using a pair of classical face and degeneracy maps of homological algebra) we always have \( C_n(R,S) \mid C_{n+1}(R,S) \) as \( R \)-\( S \)-, \( S \)-\( R \)- or \( S \)-bimodules, so that the depth \( 2n \) as well as \( 2n + 1 \) conditions coincide in the case of subring with the \( I \)-depth \( 2n \) and \( 2n + 1 \) conditions above where \( I = R \). (Note though that \( R \)-depth 1 is slightly stronger than subring depth 1 since \( R \) is not just centrally projective over \( S \) (i.e., \( R \mid S^q \) as \( S \)-bimodules) but also \( R \) is a split extension of \( S \) as \( S \)-bimodules since \( S \mid R^q \) implies \( S \mid R \); the split extension condition is satisfied by all group algebra extensions and subfactor examples of finite depth.)
Example 2.4 Let \( S \subseteq R \) be a subring pair of semisimple complex algebras. Then the minimum depth \( d(S, R) \) may be computed from the inclusion matrix \( M \), alternatively an \( n \times m \) induction-restriction table of \( n \) \( S \)-simples induced to non-negative integer linear combination of \( m \) \( R \)-simples along rows, and by Frobenius reciprocity, columns show restriction of \( R \)-simples in terms of \( S \)-simples. The procedure to obtain \( d(S, R) \) given in the paper [6] is the following: let \( M^{[2n]} = (M M^t)^n \) and \( M^{[2n+1]} = M^{[2n]} M \) (and \( M^{[0]} = I_n \)), then the matrix \( M \) has depth \( n \geq 1 \) if for some \( q \in \mathbb{Z}_+ \)

\[
M^{[n+1]} \leq qM^{[n-1]} \tag{5}
\]

The minimum depth of \( M \) is equal to \( d(S, R) \) by [1, appendix] (or Theorem 5.3 in [13] combined with [5, 6]).

In terms of the bipartite graph of the inclusion \( S \subseteq R \), \( d(S, R) \) is the lesser of the minimum odd depth and the minimum even depth [6]. The matrix \( M \) is an incidence matrix of this bipartite graph if all entries greater than 1 are changed to 1, while zero entries are retained as 0: let the \( S \)-simples be represented by \( n \) white dots in a bottom row of the graph, and \( R \)-simples by \( m \) black dots in a top row, connected by edges joining black and white dots (or not) according to the 0-1-matrix entries obtained from \( M \). The minimum odd depth of the bipartite graph is 1 plus the diameter in edges of the row of white dots (indeed an odd number), while the minimum even depth is 2 plus the largest of the diameters of the bottom row where a subset of black dots under one white dot is identified together.

Now suppose \( I \) is an ideal in \( R \). Let the primitive central idempotents of \( R \) be given by \( e_1, \ldots, e_m \) and those of \( S \) by \( f_1, \ldots, f_n \). Then \( I \) is itself a semisimple complex algebra with unit \( e = e_1 + \cdots + e_r \) (assumed with no loss of generality). Now suppose \( f_i e_j = 0 \) for \( i > s \) and all \( j \leq r \), while \( f_i e_j \neq 0 \) for \( i \leq s \) and some \( j \leq r \). Let \( J = f_1 S \oplus \cdots \oplus f_s S \), a semisimple subalgebra of \( S \): this ideal satisfies \( J \oplus \text{ann } (S I) = S \). Then it is not hard to see that \( I \)-depth of \( S \subseteq R \) is computed as the depth of the subring pair of semisimple algebras \( J \hookrightarrow I \) via \( s \mapsto es \):

\[
d^I(S, R) = d(J, I), \tag{6}
\]

the minimum depth of the \( s \times r \) submatrix \( M_1 \) in the upper lefthand corner of \( M \). This follows from the lemma where \( S/\text{ann } (S I) = J \) and the realization that \( I \otimes J \cdots \otimes J I \) is induction and restriction \( n \) times of \( I \)-simples as explained in the appendix of [1].

Example 2.5 As a sub-example of the previous example, let \( R = \mathbb{C} S_4 \), the complex group algebra of the permutation group on four letters, and \( S = \mathbb{C} S_3 \). The inclusion diagram pictured below with the degrees of the irreducible representations, is determined from the character tables of \( S_3 \) and \( S_4 \) or the branching rule (for the Young diagrams labelled by the partitions of \( n \) and representing the irreducibles of \( S_n \)).

![Inclusion Diagram](image)

This graph has minimum odd depth 5 and minimum even depth 6, whence \( d(S, R) = 5 \). Alternatively, the inclusion matrix \( M \) is given by

\[
M = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}
\]

whose bracketed powers defined above satisfy a depth 5 inequality (5).
Now let $I$ be the ideal in $R$ associated with the two-dimensional representation, the white dot labelled 2. Then $d(J, I)$ is the depth of the matrix (1), so $d^f(S, R) = 1$. If $I$ is the ideal of $R$ associated with the first three white dots in the diagram above, then $J$ is the ideal in $S$ associated to the first two black dots, and $d(J, I)$ is the minimum depth of the (upper-left hand corner) matrix

$$M' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

which has minimum depth 3. If $I$ is the ideal associated to the three white dots labelled 3,2, and 3, we similarly compute $d^f(S, R) = 4$. Finally, if $I$ is ideal associated to the first four white dots in the diagram above, the $d^f(S, R) = 5$.

**Proposition 2.6** Suppose $S \subseteq R$ is a subring pair of semisimple complex algebras and $I \subseteq R$ is an ideal. Then $d^f(S, R) \leq d(S, R)$.

**Proof.** This follows from the observation above that $d^f(S, R) = d(J, I)$ where $J \subseteq S$ and $I \subseteq R$ are both subring pairs of semisimple algebras. But $d(J, I)$ is the depth of a subgraph of the inclusion graph of $S \subseteq R$. By the description of depth of a bipartite graph as the minimum of the odd and even depths in terms of diameter of the row of black dots, it is clear that $d(J, I) \leq d(S, R)$.

The next theorem states a general case when left and right even depth of a subring below an ideal are the same; its proof is a straightforward generalization of [13, Theorem 3.4], therefore omitted. An $R$-bimodule $I$ is said to be QF relative to a subring $S \subseteq R$ below if $I_S$ and $sI$ are f.g. projectives, $s IR \simeq s \text{Hom}(I_S, S)_R$ and $RI_S \simeq R \text{Hom}(sI, S)_S$. We also suppose of the $R$-bimodule $I$ that it is a ring with multiplication, associative in all respects with the bimodule structure, such as $(x_1 \cdot r)x_2 = x_1(r \cdot x_2)$ for all $x_1, x_2 \in I, r \in R$ (called a multiplicative bimodule by R.S. Pierce). For example, an ideal $I$ in a semisimple complex algebra $R$ with semisimple subalgebra $S$ satisfies this hypothesis.

**Theorem 2.7** Suppose $I$ is a multiplicative $R$-bimodule with unit $e$ and is QF relative to a subring $S \subseteq R$. Then $S \subseteq R$ has left $I$-depth $2n$ if and only if $S \subseteq R$ has right $I$-depth $2n$.

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