A **bijection** is a one-to-one and onto mapping.
A **bijection** is a one-to-one and onto mapping.

**Example**

A mapping from set A to set B is shown in the diagram.
Introduction

**Definition**

A **bijection** is a one-to-one and onto mapping.

**Definition**

An **involution** is a bijection from a set to itself which is its own inverse.

**Example**

![Diagram showing an involution between sets A and B]
Definition

A **bijection** is a one-to-one and onto mapping.

Definition

An **involution** is a bijection from a set to itself which is its own inverse.

Example

![Diagram of bijection and involution](image)
Introduction

- The philosophy of combinatorial proof
Introduction

- The philosophy of combinatorial proof
- Bijective proof

Example

\[ \sum_{k=0}^{n} \binom{n}{k} = 2^n \]

is the number of unordered subsets of size \( k \) from a set of size \( n \).
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\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \]

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\( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) is the number of unordered subsets of size \( k \) from a set of size \( n \)

Example

Are there an even or odd number of people in the room right now?
Definition

A partition of a positive integer $n$ is an expression of $n$ as the sum of a sequence of weakly decreasing positive integers, called the parts of the partition.
Partitions

**Definition**

A partition of a positive integer \( n \) is an expression of \( n \) as the sum of a sequence of weakly decreasing positive integers, called the parts of the partition.

**Example**

There are 5 partitions of 4, as

\[
4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.
\]
Partitions

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Example

There are 5 partitions of 4, as
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4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.
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A partition \( \lambda \) of \( n \) with parts \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s \) is either written as such or in the form \( \lambda_{n_1}^{m_1} \lambda_{n_2}^{m_2} \ldots \lambda_{n_k}^{m_k} \), where each \( \lambda_{n_i} \) is a distinct part in \( \lambda \) with \( \lambda_{n_1} > \lambda_{n_2} > \ldots > \lambda_{n_k} \), and there are \( m_i \) copies of \( \lambda_{n_i} \).
**Definition**

A **partition** of a positive integer $n$ is an expression of $n$ as the sum of a sequence of weakly decreasing positive integers, called the **parts** of the partition.

**Example**

There are 5 partitions of 4, as

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$ 

A partition $\lambda$ of $n$ with parts $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s$ is either written as such or in the form $\lambda_{n_1}^{m_1} \lambda_{n_2}^{m_2} \ldots \lambda_{n_k}^{m_k}$, where each $\lambda_{n_i}$ is a distinct part in $\lambda$ with $\lambda_{n_1} > \lambda_{n_2} > \ldots > \lambda_{n_k}$, and there are $m_i$ copies of $\lambda_{n_i}$. Thus the partition $6 + 4 + 4 + 3$ of 17 could be written as $6 \geq 4 \geq 4 \geq 3$ or $6^14^23^1$. 
Definition

The Ferrers diagram of a partition $\lambda = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s$ is a diagram of left-justified boxes with $\lambda_i$ boxes in the $i$th row from the top.
Ferrers diagram

Definition

The **Ferrers diagram** of a partition $\lambda = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s$ is a diagram of left-justified boxes with $\lambda_i$ boxes in the $i$th row from the top.

Example

```
7
5
3
```
Ferrers diagram

**Definition**

The **conjugate** of a partition \( \lambda = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s \) is the partition whose Ferrers diagram has \( \lambda_i \) boxes in the \( i \)th column from the left.
**Definition**

The **conjugate** of a partition $\lambda = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s$ is the partition whose Ferrers diagram has $\lambda_i$ boxes in the $i$th column from the left.

**Example**

\[
\begin{array}{c|cccc}
  & 4 & 3 & 2 & 1 \\
\hline
 1 & & & & \\
 2 & & & & \\
 3 & & & & \\
 4 & & & &
\end{array}
\quad
\begin{array}{c|ccc}
  & 3 & 2 & 1 \\
\hline
 1 & & & \\
 2 & & & \\
 3 & & &
\end{array}
\]
Euler’s Theorem

Let $O(n)$ denote the set of partitions of $n$ into odd parts, and let $D(n)$ denote the set of partitions of $n$ into distinct parts. Then for all $n$, $|O(n)| = |D(n)|$. 

Example: $D(6): \{6, 5+1, 4+2, 3+2+1\}$; $O(6): \{5+1, 3+3, 3+1+1+1, 1+1+1+1+1+1\}$. 
Euler’s Theorem

Let $O(n)$ denote the set of partitions of $n$ into odd parts, and let $D(n)$ denote the set of partitions of $n$ into distinct parts. Then for all $n$, $|O(n)| = |D(n)|$.

Example

$D(6) : \{6, 5 + 1, 4 + 2, 3 + 2 + 1\}$
$O(6) : \{5 + 1, 3 + 3, 3 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1 + 1 + 1\}$
We give a bijection \( \phi : D(n) \rightarrow O(n) \) devised by Glaisher (1883).
Proof of Euler’s Theorem

We give a bijection $\phi : D(n) \rightarrow O(n)$ devised by Glaisher (1883). For $\lambda = \lambda_1 > \lambda_2 > \ldots > \lambda_s \in D(n)$, let $\lambda_i = 2^{p_i}\mu_i$, where $\mu_i$ is odd.
Proof of Euler’s Theorem

We give a bijection \( \phi : D(n) \to O(n) \) devised by Glaisher (1883). For \( \lambda = \lambda_1 > \lambda_2 > \ldots > \lambda_s \in D(n) \), let \( \lambda_i = 2^{p_i} \mu_i \), where \( \mu_i \) is odd. Let \( \phi(\lambda) = \mu \), where \( \mu \) has \( 2^{p_i} \) parts of \( \mu_i \), so that \( \mu \in O(n) \). Write \( \mu \) as \( \mu_{n_1}^{m_1} \mu_{n_2}^{m_2} \ldots \mu_{n_k}^{m_k} \).
We give a bijection \( \phi : D(n) \rightarrow O(n) \) devised by Glaisher (1883).

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Write \( \mu \) as \( \mu^{m_1}_{n_1} \mu^{m_2}_{n_2} \ldots \mu^{m_k}_{n_k} \). Each \( m_i = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_q} \) for exactly one sequence of nonnegative integers \( i_1 > i_2 > \ldots > i_q \).
Proof of Euler’s Theorem

We give a bijection $\phi : D(n) \rightarrow O(n)$ devised by Glaisher (1883). For $\lambda = \lambda_1 > \lambda_2 > \ldots > \lambda_s \in D(n)$, let $\lambda_i = 2^{p_i} \mu_i$, where $\mu_i$ is odd. Let $\phi(\lambda) = \mu$, where $\mu$ has $2^{p_i}$ parts of $\mu_i$, so that $\mu \in O(n)$. Write $\mu$ as $\mu_{n_1}^{m_1} \mu_{n_2}^{m_2} \ldots \mu_{n_k}^{m_k}$. Each $m_i = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_q}$ for exactly one sequence of nonnegative integers $i_1 > i_2 > \ldots > i_q$. Thus $\mu$ could only be the image under $\phi$ of some $\lambda$ with parts of the form $2^{i_t} \mu_i$, so $\phi$ is one-to-one.
Proof of Euler’s Theorem

We give a bijection \( \phi : D(n) \rightarrow O(n) \) devised by Glaisher (1883). For \( \lambda = \lambda_1 > \lambda_2 > \ldots > \lambda_s \in D(n) \), let \( \lambda_i = 2^{p_i} \mu_i \), where \( \mu_i \) is odd. Let \( \phi(\lambda) = \mu \), where \( \mu \) has \( 2^{p_i} \) parts of \( \mu_i \), so that \( \mu \in O(n) \). Write \( \mu \) as \( \mu_{m_1}^{n_1} \mu_{m_2}^{n_2} \ldots \mu_{m_k}^{n_k} \). Each \( m_i = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_q} \) for exactly one sequence of nonnegative integers \( i_1 > i_2 > \ldots > i_q \). Thus \( \mu \) could only be the image under \( \phi \) of some \( \lambda \) with parts of the form \( 2^{i_t} \mu_i \), so \( \phi \) is one-to-one. Since such a \( \lambda \) must have distinct parts, we have a well-defined inverse mapping \( \phi^{-1} : O(n) \rightarrow D(n) \), so \( \phi \) is onto.
Fine’s Theorem

Let $Q(n)$ be the set of partitions of $n$ into distinct parts where the smallest part is odd. Then $|Q(n)|$ is odd if and only if $n$ is a perfect square.

Example:

$Q(6): \{5 + 1, 3 + 2 + 1\}$

$Q(7): \{7, 6 + 1, 4 + 3, 4 + 2 + 1\}$

$Q(9): \{9, 8 + 1, 6 + 3, 6 + 2 + 1, 5 + 3 + 1\}$

$Q(10): \{9 + 1, 7 + 3, 7 + 2 + 1, 6 + 3 + 1, 5 + 4 + 1, 4 + 3 + 2 + 1\}$
Fine’s Theorem

**Theorem**

Let $Q(n)$ be the set of partitions of $n$ into distinct parts where the smallest part is odd. Then $|Q(n)|$ is odd if and only if $n$ is a perfect square.
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Let $Q(n)$ be the set of partitions of $n$ into distinct parts where the smallest part is odd. Then $|Q(n)|$ is odd if and only if $n$ is a perfect square.

**Example**

- $Q(6)$: $\{5 + 1, 3 + 2 + 1\}$
- $Q(7)$: $\{7, 6 + 1, 4 + 3, 4 + 2 + 1\}$
- $Q(9)$: $\{9, 8 + 1, 6 + 3, 6 + 2 + 1, 5 + 3 + 1\}$
- $Q(10)$: $\{9+1, 7+3, 7 + 2 + 1, 6 + 3 + 1, 5 + 4 + 1, 4 + 3 + 2 + 1\}$
Theorem

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Idea of the proof: Define an involution $\kappa : Q(n) \rightarrow Q(n)$ which has exactly one fixed point if and only if $n$ is a perfect square (and has no fixed points if and only if $n$ is not a perfect square).
Proof of Fine’s Theorem

Let $\lambda \in Q(n)$, with odd parts $s_k > s_{k-1} > \ldots > s_1$ and smallest even part $e_1$. 
Proof of Fine’s Theorem

Let $\lambda \in Q(n)$, with odd parts $s_k > s_{k-1} > \ldots > s_1$ and smallest even part $e_1$. If there are no even parts, define $e_1 = \infty$. 
Proof of Fine’s Theorem

Let $\lambda \in Q(n)$, with odd parts $s_k > s_{k-1} > \ldots > s_1$ and smallest even part $e_1$. If there are no even parts, define $e_1 = \infty$. Define $t_i = s_i - (2i - 1)$, for $i \leq k$, and let $t_{k+1} = \infty$. 
Proof of Fine’s Theorem

Let $\lambda \in Q(n)$, with odd parts $s_k > s_{k-1} > \ldots > s_1$ and smallest even part $e_1$. If there are no even parts, define $e_1 = \infty$. Define $t_i = s_i - (2i - 1)$, for $i \leq k$, and let $t_{k+1} = \infty$. Let $\alpha$ be the smallest $i$ such that $t_i > 0$. 

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Now, we construct $\kappa(\lambda)$ by performing one of the following three actions, depending on the case.
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Now, we construct $\kappa(\lambda)$ by performing one of the following three actions, depending on the case.

1. If $t_\alpha < e_1$, then split $s_\alpha$ into $2\alpha - 1$ and $t_\alpha$. 
Proof of Fine’s Theorem

Let $\lambda \in Q(n)$, with odd parts $s_k > s_{k-1} > \ldots > s_1$ and smallest even part $e_1$. If there are no even parts, define $e_1 = \infty$. Define $t_i = s_i - (2i - 1)$, for $i \leq k$, and let $t_{k+1} = \infty$. Let $\alpha$ be the smallest $i$ such that $t_i > 0$.

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1. If $t_\alpha < e_1$, then split $s_\alpha$ into $2\alpha - 1$ and $t_\alpha$.
2. If $t_\alpha \geq e_1$, and $e_1 < \infty$, then combine $s_{\alpha-1}$ with $e_1$. 
Proof of Fine’s Theorem

Let $\lambda \in Q(n)$, with odd parts $s_k > s_{k-1} > \ldots > s_1$ and smallest even part $e_1$. If there are no even parts, define $e_1 = \infty$. Define $t_i = s_i - (2i - 1)$, for $i \leq k$, and let $t_{k+1} = \infty$. Let $\alpha$ be the smallest $i$ such that $t_i > 0$.

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1. If $t_\alpha < e_1$, then split $s_\alpha$ into $2\alpha - 1$ and $t_\alpha$.
2. If $t_\alpha \geq e_1$, and $e_1 < \infty$, then combine $s_{\alpha-1}$ with $e_1$.
3. If $t_\alpha = e_1 = \infty$, then do nothing.
Proof of Fine’s Theorem
Proof of Fine’s Theorem
Proof of Fine’s Theorem
Proof of Fine’s Theorem
Proof of Fine’s Theorem
Proof of Fine’s Theorem
Proof of Fine’s Theorem
Kolberg’s Theorem

The partition function $p(n)$ takes infinitely many even and odd values.
Proof of Kolberg’s Theorem

Let $SC(n)$ denote the set of self-conjugate partitions of $n$. 

Thus $p(n)$ has the same parity as $|DO(n)|$. 

Let $DO(n) = D(n) \cap O(n)$ be the set of partitions of $n$ into distinct odd parts.

Then $|SC(n)| = |DO(n)|$. 

Thus $p(n)$ has the same parity as $|DO(n)|$. 

Proof of Kolberg’s Theorem

Let $SC(n)$ denote the set of self-conjugate partitions of $n$. Then $p(n)$ has the same parity as $|SC(n)|$. 

\[
\begin{array}{cccc}
5 & & & \\
3 & & & \\
2 & & & \\
1 & & & \\
1 & & & \\
\end{array}
\]
Proof of Kolberg’s Theorem

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Let $SC(n)$ denote the set of self-conjugate partitions of $n$. Then $p(n)$ has the same parity as $|SC(n)|$.

Let $DO(n) = D(n) \cap O(n)$ be the set of partitions of $n$ into distinct odd parts. Then $|SC(n)| = |DO(n)|$. 

\[
\begin{array}{ccccccccc}
5 & & & & & & & & \\
3 & & & & & & & & \\
2 & & & & & & & & \\
1 & & & & & & & & \\
1 & & & & & & & & \\
\end{array}
\]

\[
\begin{array}{cccccccc}
5 & & & & & & & & \\
3 & & & & & & & & \\
2 & & & & & & & & \\
1 & & & & & & & & \\
1 & & & & & & & & \\
3 & & & & & & & & \\
9 & & & & & & & & \\
\end{array}
\]
Proof of Kolberg’s Theorem

Let $SC(n)$ denote the set of self-conjugate partitions of $n$. Then $p(n)$ has the same parity as $|SC(n)|$.

Let $DO(n) = D(n) \cap O(n)$ be the set of partitions of $n$ into distinct odd parts. Then $|SC(n)| = |DO(n)|$.

Thus $p(n)$ has the same parity as $|DO(n)|$. 
Proof of Kolberg’s Theorem

For any $\lambda \in DO(n)$ denote the parts of $\lambda$ by $\lambda_1 > \lambda_2 > \ldots > \lambda_s$. 

Thus $|DO(n)|$ differs in parity from $|DO(n+1)|$ precisely when $|DO_1(n+1)|$ is odd.

Thus to prove the theorem we can show that $|DO_1(n)|$ is odd for infinitely many $n$. 
Proof of Kolberg’s Theorem

For any $\lambda \in DO(n)$ denote the parts of $\lambda$ by $\lambda_1 > \lambda_2 > \ldots > \lambda_s$. Let $DO_1(n) = \{\lambda \in DO(n)|\lambda_1 - \lambda_2 = 2, \lambda_s > 1\}$. 

Thus $|DO(n)|$ differs in parity from $|DO(n+1)|$ precisely when $|DO_1(n+1)|$ is odd. Thus to prove the theorem we can show that $|DO_1(n)|$ is odd for infinitely many $n$. 
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Then $|DO(n)| = |DO(n + 1) \setminus DO_1(n + 1)|$. 

Thus $|DO(n)|$ differs in parity from $|DO(n + 1)|$ precisely when $|DO_1(n + 1)|$ is odd. Thus to prove the theorem we can show that $|DO_1(n)|$ is odd for infinitely many $n$. 
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Thus to prove the theorem we can show that $|DO_1(n)|$ is odd for infinitely many $n$. 
Proof of Kolberg’s Theorem

Let $DO_j(n) = \{ \lambda \in DO(n) | \lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = \ldots = \lambda_j - \lambda_{j+1} = 2, \lambda_s > 1 \}$. 

\begin{align*}
11 & 9 & 7 & 3 \\
10 & 8 & 6 & 2 \\
9 & 7 & 5 & 1 \\
8 & 6 & 4 & \\
7 & 5 & 3 & \\
6 & 4 & 2 & \\
5 & 3 & 1 & \\
4 & 2 & & \\
3 & 1 & & \\
2 & & & \\
1 & & & \\
\end{align*}
Proof of Kolberg’s Theorem

Let $DO_j(n) = \{ \lambda \in DO(n) \mid \lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = \ldots = \lambda_j - \lambda_{j+1} = 2, \lambda_s > 1 \}$.

Then for all positive integers $j$, 
$|DO_j(n)| = |DO_j(n + 2j + 2) \setminus DO_{j+1}(n + 2j + 2)|$.

$\phi : DO_2(24) \rightarrow DO_2(30) \setminus DO_3(30)$
Proof of Kolberg’s Theorem

Let $DO_j(n) =$
$\{ \lambda \in DO(n) | \lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = \ldots = \lambda_j - \lambda_{j+1} = 2, \lambda_s > 1 \}$. 

Then for all positive integers $j$,
$|DO_j(n)| = |DO_j(n + 2j + 2) \setminus DO_{j+1}(n + 2j + 2)|$.

Thus $|DO_j(n)|$ differs in parity from $|DO_j(n + 2j + 2)|$ precisely when $|DO_{j+1}(n + 2j + 2)|$ is odd.
Given $k \geq 1$, we construct $n > k$ such that $|DO_1(n)|$ is odd.
Given \( k \geq 1 \), we construct \( n > k \) such that \(|DO_1(n)|\) is odd. Note that 
\[
DO_{k-2}(k^2 - 1) = \{(2k - 1) + (2k - 3) + \ldots + 3\}
\]
and therefore 
\[
|DO_{k-2}(k^2 - 1)| = 1.
\]
Proof of Kolberg’s Theorem

Given $k \geq 1$, we construct $n > k$ such that $|DO_1(n)|$ is odd. Note that $DO_{k-2}(k^2 - 1) = \{(2k - 1) + (2k - 3) + \ldots + 3\}$ and therefore $|DO_{k-2}(k^2 - 1)| = 1$. Thus $|DO_{k-3}(k^2 - 1)|$ and $|DO_{k-3}(k^2 - 1 - 2(k - 2))|$ differ in parity, so we can pick $n_{k-3} \in \{k^2 - 1, k^2 - 1 - 2(k - 2)\}$ such that $|DO_{k-3}(n_{k-3})|$ is odd.
Proof of Kolberg’s Theorem

Given \( k \geq 1 \), we construct \( n > k \) such that \( |DO_1(n)| \) is odd. Note that \( DO_{k-2}(k^2 - 1) = \{(2k - 1) + (2k - 3) + \ldots + 3\} \) and therefore \( |DO_{k-2}(k^2 - 1)| = 1 \). Thus \( |DO_{k-3}(k^2 - 1)| \) and \( |DO_{k-3}(k^2 - 1 - 2(k - 2))| \) differ in parity, so we can pick \( n_{k-3} \in \{k^2 - 1, k^2 - 1 - 2(k - 2)\} \) such that \( |DO_{k-3}(n_{k-3})| \) is odd. Then \( |DO_{k-4}(n_{k-3})| \) and \( |DO_{k-4}(n_{k-3} - 2(k - 3))| \) differ in parity, so we can pick \( n_{k-4} \in \{n_{k-3}, n_{k-3} - 2(k - 3)\} \) such that \( |DO_{k-4}(n_{k-4})| \) is odd.
Proof of Kolberg’s Theorem

Given $k \geq 1$, we construct $n > k$ such that $|DO_1(n)|$ is odd. Note that $DO_{k-2}(k^2 - 1) = \{(2k - 1) + (2k - 3) + \ldots + 3\}$ and therefore $|DO_{k-2}(k^2 - 1)| = 1$. Thus $|DO_{k-3}(k^2 - 1)|$ and $|DO_{k-3}(k^2 - 1 - 2(k - 2))|$ differ in parity, so we can pick $n_{k-3} \in \{k^2 - 1, k^2 - 1 - 2(k - 2)\}$ such that $|DO_{k-3}(n_{k-3})|$ is odd. Then $|DO_{k-4}(n_{k-3})|$ and $|DO_{k-4}(n_{k-3} - 2(k - 3))|$ differ in parity, so we can pick $n_{k-4} \in \{n_{k-3}, n_{k-3} - 2(k - 3)\}$ such that $|DO_{k-4}(n_{k-4})|$ is odd. We iterate this process to find $n_1$. Then $|DO_1(n_1)|$ is odd, and

$$n_1 \geq k^2 - 1 - (2(k - 2) + 2(k - 3) + \ldots + 4) = k^2 - 1 - 2(2 + 3 + \ldots + k - 2) = k^2 - 1 - k(k - 3) = 3k - 1 > k.$$
Special thanks to our advisor Mark Krusemeyer.

Sources and Suggested Reading:

