Problem: Show that when $n$ is a positive integer,

$$
\sum_{k \geq 0} \binom{n}{k} \binom{2k}{k} = \sum_{k \geq 0} \binom{n}{2k} \binom{2k}{k} 3^{n-2k}.
$$

Solution I by CMC 328, Carleton College, Northfield, MN. Consider a country with $n$ states, in which each state has a senior senator and a junior senator (distinguishable). Each senator is Democratic (D), Republican (R), or Independent (I). States are either independent (both senators Independent) or partisan (neither senator Independent); no state has exactly one Independent senator. A balanced senate is an assignment of parties to the $2n$ senators so that the number of Republican senators equals the number of Democratic senators. We show that both sides of the equation count the balanced senates.

Left side: There are $\binom{n}{k}$ ways to choose $k$ states to be partisan. There are $\binom{2k}{k}$ ways to assign their senators to parties. Thus the number of balanced senates is

$$
\sum_{k \geq 0} \binom{n}{k} \binom{2k}{k}.
$$

Right side: The number of states with two Republicans equals the number with two Democrats; let this number be $k$. There are $\binom{n}{2k}$ ways to choose these states, and there are $\binom{2k}{k}$ ways to assign their senators to parties. Each remaining state is R/D, D/R, or I/I; these can be assigned in $3^{n-2k}$ ways. Counted this way, the number of balanced senates is

$$
\sum_{k \geq 0} \binom{n}{2k} \binom{2k}{k} 3^{n-2k}.
$$

[...]

CMC 328 also observed that for senates with $2m$ more Democrats than Republicans, the identity generalizes to

$$
\sum_{k \geq 0} \binom{n}{k} \binom{2k}{k + m} = \sum_{k \geq 0} \binom{n}{2k + m} \binom{2k + m}{k} 3^{n-(2k+m)}.
$$

To allow $p$ independent parties (with each independent state having two senators of the same independent party), multiply the summand on the left by $p^{n-k}$, and change 3 to $2 + p$ on the right.