A New Fibonacci Identity

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### Definition

A **tiling** of a board of length $n$ and height 1 consists of a non-overlapping placement of squares ($1 \times 1$) and dominoes ($2 \times 1$) which completely cover the board.
Tiling a Board

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Example
Tiling a Board

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Example

[Diagram showing a board with tiles and dominoes placed in a tiling pattern]
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Example
## Counting Board Tilings

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of tilings</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
</tr>
</tbody>
</table>
There are $F_n$ ways to tile a board of length $n$ with squares and dominoes, where $F_0 = F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. 
Why Tilings are Cool

Proof.

How many ways are there to tile a board of length $2n$?

Answer 1: $F_{2n}$

Answer 2: $(F_n)^2 + (F_n - 1)^2$
Why Tilings are Cool

Theorem

\[ F_{2n} = (F_n)^2 + (F_{n-1})^2, \text{ for all } n \geq 1. \]
Why Tilings are Cool

**Theorem**

\[ F_{2n} = (F_n)^2 + (F_{n-1})^2, \text{ for all } n \geq 1. \]

**Example**

When \( n = 4 \), we have \( F_{2n} = F_8 = 34 \) and

\[ (F_n)^2 + (F_{n-1})^2 = (F_4)^2 + (F_3)^2 = 5^2 + 3^2 = 34. \]
Why Tilings are Cool

**Theorem**

\[ F_{2n} = (F_n)^2 + (F_{n-1})^2, \text{ for all } n \geq 1. \]

**Proof.**

How many ways are there to tile a board of length \( 2n \)?

Answer 1: \( F_{2n} \)

Answer 2: \( (F_n)^2 + (F_{n-1})^2 \)
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**Theorem**

\[ F_{2n} = (F_n)^2 + (F_{n-1})^2, \text{ for all } n \geq 1. \]

**Proof.**

How many ways are there to tile a board of length 2\(n\)?

Answer 1: \(F_{2n}\)
Why Tilings are Cool

Theorem

\[ F_{2n} = (F_n)^2 + (F_{n-1})^2, \text{ for all } n \geq 1. \]

Proof.

How many ways are there to tile a board of length \(2n\)?

Answer 1: \(F_{2n}\)

Answer 2:
Theorem

\[ F_{2n} = (F_n)^2 + (F_{n-1})^2, \text{ for all } n \geq 1. \]

Proof.

How many ways are there to tile a board of length 2n?

Answer 1: \( F_{2n} \)

Answer 2: \( (F_n)^2 + (F_{n-1})^2 \)
Theorem (Lonoff, Ostroff)

\[ \sum_{k=1}^{n} F_{2k-4} 2^{n-k} = F_{2n-1}, \text{ for all } n \geq 1. \]
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How many ways are there to tile a board of length $2n - 1$?
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Theorem (Lonoff, Ostroff)

\[ \sum_{k=1}^{n} F_{2k-4}2^{n-k} = F_{2n-1}, \text{ for all } n \geq 1. \]

How many ways are there to tile a board of length 2\(n - 1\)?
Answer 1: \(F_{2n-1}\)
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Theorem (Lonoff, Ostroff)

\[ \sum_{k=1}^{n} F_{2k-4} 2^{n-k} = F_{2n-1}, \text{ for all } n \geq 1. \]

How many ways are there to tile a board of length $2n - 1$?

Answer 1: $F_{2n-1}$

Answer 2:
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Theorem (Lonoff, Ostroff)

\[ \sum_{k=1}^{n} F_{2k-4}2^{n-k} = F_{2n-1}, \text{ for all } n \geq 1. \]

How many ways are there to tile a board of length \(2n - 1\)?

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Answer 2:
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Theorem (Lonoff, Ostroff)

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How many ways are there to tile a board of length \(2n - 1\)?

Answer 1: \( F_{2n-1} \)

Answer 2: \( \sum_{k=1}^{n} F_{2k-4}2^{n-k} \)
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Theorem (Lonoff, Ostroff)

\[ \sum_{k=1}^{n} F_{2k-4}2^{n-k} = F_{2n-1}, \text{ for all } n \geq 1. \]

This naturally generalizes to

\[ F_{mn+r} = F_r F_m^n + \sum_{k=1}^{n} F_{mk-m+r-1} F_{m-1} F_{m}^{n-k} \]
Acknowledgements

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Thanks for your attention!