Notes for Graduate Student Seminar: Lang’s Thesis

Man Cheung Tsui

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Abstract

The Brauer group $\text{Br}(k)$ of a field $k$ is an important object of study in number theory. A convenient condition to force the Brauer group to be trivial is for $k$ to be a quasi-algebraically closed field. Quasi-algebraically closed fields are part of a more general notion of $C_1$ fields, introduced in Lang’s thesis in 1951. The goal of this talk is to prove that finite fields $\mathbb{F}_q$ and function fields $\bar{k}(t)$ over algebraically closed fields are quasi-algebraically closed hence have trivial Brauer groups.

Motivation

Representation of groups as matrices. To understand a finite group $G$, we can act it on a finite-dimensional vector space $V$, so that $G$ is now represented by matrices\(^1\) in $\text{GL}(V)$. Two representations $G \to \text{GL}(V)$ and $G \to \text{GL}(W)$ give rise to a third representation $G \to \text{GL}(V \oplus W)$ by:

$$g \leftrightarrow A \in \text{GL}(V) \quad g \leftrightarrow B \in \text{GL}(W) \quad \Rightarrow \quad g \leftrightarrow \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \text{GL}(V \oplus W).$$

Conversely, given a representation $G \to \text{GL}(V)$, we want to decompose it into irreducible blocks\(^2\). This is provided by:

**Theorem 1** (Maschke, Artin-Wedderburn). Let $G$ be a finite group. Let $k$ be a field whose characteristic does not divide $|G|$. Then:

1. The group ring $k[G]$ is isomorphic to the product of matrix rings $M_n(D_i)$ where $D_i$ are division algebras that are finite-dimensional over $k$.

2. The action of $G$ on these $k$-vector spaces $M_n(D_i)$ are the irreducible representations of $G$.

**Example.** The finite-dimensional division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. Consider $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}]$. This is a 3-dimensional $\mathbb{R}$-vector space, so either:

- $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}] \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ or

\(^1\) Upon a choice of basis

\(^2\) These are called the irreducible representations.
\begin{itemize}
  \item $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}] \cong \mathbb{R} \times \mathbb{C}$.
\end{itemize}

But $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}]$ has only one $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}]$-invariant subspace of dimension 1 so $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}] \cong \mathbb{R} \times \mathbb{C}$. Thus $\mathbb{R}[\mathbb{Z}/3\mathbb{Z}]$ has two irreducible representations over $\mathbb{R}$.

Therefore, to classify the irreducible representations over $k$, one first classifies the finite-dimensional division algebras over $k$. We store this information as follows.

Define the **Brauer group of a field** $k$ to be the set of (isomorphism classes of) division algebras which are finite-dimensional over $k$ and have center $k$. The Brauer group of $k$ is denoted $\text{Br}(k)$.

For example, $\text{Br}(\mathbb{C}) = \{\mathbb{C}\}$ and $\text{Br}(\mathbb{R}) = \{\mathbb{R}, \mathbb{H}\}$.

We define the group structure on $\text{Br}(k)$ in the following manner. Given two division algebras $D_1, D_2 \in \text{Br}(k)$, it turns out that $D_1 \otimes_k D_2 \cong M_d(D_3)$ for some $D_3 \in \text{Br}(k)$. Thus define the group multiplication in $\text{Br}(k)$ by $[D_1] \cdot [D_2] = [D_3]$.

**The Brauer group under field extension.** Let $k'/k$ be a field extension. Given a division algebra $D \in \text{Br}(k)$, the tensor product $D \otimes_k k'$ is isomorphic to some matrix ring $M_d(D')$ with $D' \in \text{Br}(k')$. There are two consequences to this.

First we have $D \otimes_k \bar{k} \cong M_d(\bar{k})$. This can be improved:\footnote{See Corollary 2.2.6 in [1].} there exists a Galois field extension $k'/k$ for which $D \otimes_k k' \cong M_d(k')$.\footnote{Such a field $k'$ is said to split the division algebra $D$.}

Second, $\text{Br}(-)$ can be viewed as a functor $\text{Fields} \to \text{Groups}$ taking a morphism of fields $k \hookrightarrow k'$ to

$$\text{Br}(k) \to \text{Br}(k')$$

$$[D] \mapsto [D'],$$

where $D'$ is the division algebra for which $D \otimes_k k' \cong M_d(D')$. Thus $\text{Br}(-)$ is a functorial way to tell fields apart.

**Finite-dimensional division algebra over its center has square dimension.** Let $D \in \text{Br}(k)$ be a division algebra. We claim that $\dim_k D$ is a square number. Under the natural injection $D \hookrightarrow D \otimes_k \bar{k} \cong M_d(\bar{k})$, a $k$-basis $\{\omega_1, \ldots, \omega_r\}$ for $D$ remains a $\bar{k}$-basis for $D \otimes_k \bar{k}$. Therefore:

$$\dim_k D = \dim_{\bar{k}}(D \otimes_k \bar{k}) = \dim_{\bar{k}} M_d(\bar{k}) = d^2.$$

**$C_i$ Fields**

Another way to tell fields apart is by a property. Define a field $k$ to be:
• $C_0$ if any degree $d$ form over $k$ in $n > 1$ variables has a nontrivial zero.
• $C_1$ if any degree $d$ form over $k$ in $n > d^2$ variables has a nontrivial zero.
• $C_i$ if any degree $d$ form over $k$ in $n > d^i$ variables has a nontrivial zero.

One can show that a field is $C_0$ if and only if it is an algebraically closed field. Properties about $C_1$ fields were discussed and known by Artin, Tsen, and others under the label quasi-algebraically closed fields. Lang generalized this to the notion of $C_i$ fields.

**Proposition 2.** If a field $k$ is $C_1$, then $Br(k)$ is trivial.

*Proof.* We prove the contrapositive. Let $D \in Br(k)$ be a division algebra of dimension $d^2 > 1$ over $k$. Let $\{\omega_1, \ldots, \omega_{d^2}\}$ be a $k$-basis for $D$. Let $k'/k$ be a Galois field extension which splits $D$ i.e. $D \otimes_k k' \cong M_d(k')$. Then the injective $k$-algebra homomorphism

$$\varphi : D \hookrightarrow D \otimes_k k' \cong M_d(k')$$

sends a generic element $x = x_1 \omega_1 + \cdots + x_{d^2} \omega_{d^2} \in D$ to a $d \times d$ matrix $[x]$ over $k'$. Consider the polynomial

$$g(x_1, \ldots, x_{d^2}) = \det([x])$$

in the variables $x_1, \ldots, x_{d^2}$.

Let $A_i = [\omega_i]$ so $g(x_1, \ldots, x_{d^2}) = \det(x_1 A_1 + \cdots + x_{d^2} A_{d^2}).$

**Claim 1.** $g$ is a homogeneous polynomial of degree $d$.

*Proof.* Clearly $g(\lambda x_1, \ldots, \lambda x_{d^2}) = \lambda^d g(x_1, \ldots, x_{d^2}).$

**Claim 2.** $g$ depends on all the variables $x_1, \ldots, x_{d^2}$.

*Proof.* We have $g(0, 0, \ldots, 0) = 0$ and $g(1, 0, \ldots, 0) = \det A_1 \neq 0$ since $\omega_1 \in D$ is invertible and $\varphi$ is a $k$-algebra homomorphism. So $g$ depends on the variable $x_1$. Similarly for $x_i$.

**Claim 3.** $g \in k[x_1, \ldots, x_{d^2}]$.

*Proof.* Let $\sigma \in \text{Gal}(k'/k)$. By Skolem-Noether,

$$\sigma \cdot g(x) = \det(x_1 \sigma(A_1) + \cdots + \sigma(A_{d^2})) = \det(x_1 B A_1 B^{-1} + \cdots + x_{d^2} B A_{d^2} B^{-1}) = g(x)$$

for some $B \in GL_d(k')$.

If $x \neq 0$, $\det([x]) \neq 0$ so $g$ only has the trivial zero. Therefore $k$ is not $C_1$. □
**C_i fields: examples**

We now see that certain fields are $C_1$, hence have trivial Brauer group. This section follows [4], section 3 of [3], and chapter 3 of [2] verbatim.

**Proposition 3.** Finite field $\mathbb{F}$ is $C_1$.

*Proof.* Let $|\mathbb{F}| = q = p^k$. Let $f(x_1, ..., x_n)$ be a degree $d$ form over $\mathbb{F}$ with $n > d$. We show that $f$ has a nontrivial zero $x \in \mathbb{F}^n$.

If $i < q - 1$ then

$$\sum_{x \in \mathbb{F}} x^i = 0$$

so the sum over $\mathbb{F}^n$ of any polynomial in $x_1, ..., x_n$ of degree less than $n(q - 1)$ also vanishes.

We have the indicator function

$$1 - f(x)^{q-1} = \begin{cases} 1 & \text{if } x \text{ is a zero of } f, \\ 0 & \text{else.} \end{cases}$$

Thus

$$\# \text{ zeros of } f = \sum_{x \in \mathbb{F}^n} (1 - f(x)^{q-1}) \equiv 0 \pmod{p}.$$ 

As $0 \in \mathbb{F}^n$ is a zero of $f$, by above, $f$ has a nontrivial zero. \qed

Our goal is now to show $\bar{\mathbb{k}}(t)$ is $C_1$. This requires introducing the concept of a normic form and a key theorem due to Lang and Nagata.

Let $K/k$ be a finite field extension of degree $n > 1$ and consider a $k$-basis $\{\omega_1, ..., \omega_n\}$ for $K$. A generic element $x \in K$ can be written as

$$x = x_1\omega_1 + \cdots + x_n\omega_n.$$ 

Consider the left multiplication map $l_x : K \to K : y \mapsto xy$. As a $k$-linear map, $l_x$ has a matrix representation $[l_x]$ with respect to our chosen basis. Its determinant $\det[l_x]$ is then a homogeneous polynomial of degree $n$ in the $n$ variables $x_1, ..., x_n$. This polynomial is called the **norm form of** $K/k$.

**Example.** Consider the field extension $\mathbb{C}/\mathbb{R}$. Viewing $\mathbb{C}$ as an $\mathbb{R}$-vector space, the map given by multiplication by $x + iy$ has the matrix representation

$$[k + iy] = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

in the $\mathbb{R}$-basis $\{1, i\}$. Therefore the norm form for $\mathbb{C}$ over $\mathbb{R}$ is

$$\det \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = x^2 + y^2.$$
More generally, a **normic form over a field** \( k \) is a homogeneous polynomial \( \phi \) over \( k \) of degree \( d \) in \( n \) variables such that \( n = d \) and \( \phi \) has only the trivial zero over \( k \).

**Example.** For the field extension \( \mathbb{C}/\mathbb{R} \), we can use the norm form \( N(x, y) = x^2 + y^2 \) to generate more normic forms by plugging \( N \) into itself. For example,

\[
N(N(x_1, x_2), N(y_1, y_2)) = (x_1^2 + x_2^2)^2 + (y_1^2 + y_2^2)^2
\]

is a normic form. Therefore:

**Lemma 4.** Let \( k \) be a field that is not algebraically closed. Then there exist normic forms over \( k \) of arbitrarily large degree.

**Theorem 5** (Lang-Nagata). Let \( k \) be a \( C_i \) field. Let \( f_1, ..., f_r \) be homogeneous polynomials over \( k \) of degree \( d \) in \( n \) variables. If \( n > rd^i \), then they have a nontrivial common zero in \( k^n \).

**Proof.** The \( i = 0 \) case corresponds to \( k \) being algebraically closed. This is handled by classical algebraic geometry; see Proposition 7.1 of Chapter 1 in Hartshorne.

Therefore, let us now assume that \( k \) is not algebraically closed.

Let \( i = 1 \). Let \( \phi \) be a normic form of degree \( e \geq r \). Consider the form

\[
\phi^{(1)} = \phi(f_1, ..., f_r | f_1, ..., f_r | f_1, ..., f_r | 0, ..., 0)
\]

where after each slash we use new variables, and we insert as many complete sets of \( f \)'s as possible.

Thus \( \phi^{(1)} \) has \( n \left\lceil \frac{e}{r} \right\rceil \) variables and has degree \( de \leq dr \left( \left\lceil \frac{e}{r} \right\rceil + 1 \right) \). If \( k \) is \( C_1 \), we want

\[
n \left\lceil \frac{e}{r} \right\rceil > dr \left( \left\lceil \frac{e}{r} \right\rceil + 1 \right)
\]

or

\[
(n - dr) \left\lceil \frac{e}{r} \right\rceil > dr.
\]

Since \( k \) is \( C_1 \), \( n - dr > 0 \), so the above inequality holds for large enough \( e \). Then \( \phi^{(1)} \) has a nontrivial zero. Since \( \phi \) is normic, it is a common zero of all the \( f \)'s.

For \( i > 1 \), see Theorem 3.3.7 of [3] or Theorem 3.4 of [2].

Two consequences of this theorem are:

**Theorem 6.** If \( k \) is a \( C_i \) field, then every algebraic extension of \( k \) is also a \( C_i \) field. **Consequence 1 of Lang-Nagata**
Proof. (Lang) It suffices to prove the theorem for a finite extension $K$ of $k$, since the coefficients of any given form lies in a finite extension.

Let $K$ be a finite field extension of $k$ of degree $e$. Let $\{\omega_1, ..., \omega_e\}$ be a $k$-basis of $K$. Let $f(\bar{x})$ be a form over $K$ of degree $d$ and in $n > d^i$ variables. Introduce new variables $\bar{x}_{ij}$ with respect to our basis, such that

$$x_i = \bar{x}_{i0}\omega_1 + \cdots + \bar{x}_{ie}\omega_e$$

for $i = 1, ..., n$. Then

$$f(\bar{x}) = f_1(\bar{x})\omega_1 + \cdots + f_e(\bar{x})\omega_e$$

for some forms $f_i$ of degree $d^i$ in $n$ variables. Since $en > ed^i$, by the previous theorem, these forms $f_i$ have a nontrivial common zero $\bar{x}$ over $K$. This gives a nontrivial zero $x$ for $f$ over $K$.

Next we have

**Theorem 7.** If $k$ is a $C_i$ field, and $K$ is an extension of $k$ of finite transcendence degree $j$, then $K$ is a $C_{i+j}$ field.

**Proof.** (Tsen, essentially) By the previous theorem, we are reduced to purely transcendental extensions. By induction on $j$, we are reduced to the case $K = k(t)$. By homogeneity, it suffices to consider forms with coefficients in the polynomial ring $k[t]$.

Let $f$ be a form of degree $d$ in $n > d^{i+1}$ variables with coefficients in $k[t]$. Introduce new variables $\bar{x}_{ij}$ with

$$x_i = \bar{x}_{i0} + \bar{x}_{i1}t + \bar{x}_{i2}t^2 + \cdots + \bar{x}_{is}t^s$$

where $s$ will be chosen later. If $r$ is the highest degree of the coefficients of $f$, we get

$$f(\bar{x}) = f_0(\bar{x}) + f_1(\bar{x})t + \cdots + f_{ds+r}(\bar{x})t^{ds+r}$$

where each form $f_i$ is of degree $d$ in $n(s + 1)$ variables.

We finish by applying Theorem 5 to these forms $f_i$. For large enough $s$, the inequality

$$n(s + 1) > d^i(ds + r + 1)$$

holds, since this is the same as

$$(n - d^{i+1})s > d^i(r + 1) - n.$$ 

Therefore the $f_i$ have a nontrivial common zero. This gives a nontrivial zero to $f$ over $k(T)$.

As a corollary, since $\bar{k}$ is a $C_0$ field, we deduce that $\bar{k}(t)$ is a $C_1$ field. This is the original Tsen’s theorem.
Final remarks

**Lemma 8.** Let $k$ be a field complete under a non-archimedean valuation and let its ring of integers $\mathfrak{o}$ be compact. Let $f$ be a form over $F$ and suppose that there exists a sequence of forms $f_i$ converging to $f$ such that each $f_i$ has a nontrivial zero in $k$. Then $f$ also has a nontrivial zero in $k$.

*Proof.* We may assume all coefficients and zeros are integers i.e. elements of $\mathfrak{o}$. By homogeneity, we may assume that some $\alpha_i$ in a zero $(\alpha_1, ..., \alpha_n)$ of $f_i$ is a unit. We obtain in this way a collection of $n$-tuples in the Cartesian product $\mathfrak{o} \times \cdots \times \mathfrak{o}$. This product is compact, so our collection has an accumulation point which will be our desired zero. It is nontrivial since each element of our collection has at least one unit component.

**Theorem 9.** Let $F$ be a finite field. Then $F((t))$ is a $C_2$ field.

*Proof.* Given the form $f(x_1, ..., x_n)$ of degree $d$, $n > d^2$ with integral coefficients, we must show it has a nontrivial zero. If we omit the coefficients of $f$ after a power of $t$, we obtain a form over $F[t]$. As the field $F(t)$ is $C_2$, this form has a nontrivial zero. Doing this for each power of $t$ we obtain an approximating sequence of forms, each having a zero. We can apply the above lemma to complete the proof.

Following the analogy between function fields and number fields, Artin conjectured that $\mathbb{Q}_p$ is $C_2$. This turned out to be false: Guy Terjanian constructed an explicit counterexample over $\mathbb{Q}_2$ in 1966. However, Ax and Kochen managed to prove the following surprising statement: for any integer $d > 0$, there exists a finite set of primes $P(d)$ such that the $C_2$ property holds for forms of degree $d$ in $\mathbb{Q}_p$ for all $p \notin P(d)$.

The proof of the Ax-Kochen theorem uses model theory. For more on this, see [3].

**References**

