Problem: Consider the function

\[ f(x) = \begin{cases} 
  \frac{x^2}{\sqrt{2ax^2+1} - \sqrt{ax^2+1}} & x < 0 \\
  1 & x = 0 \\
  \frac{a}{x} - \frac{2a}{x^2 + x} & x > 0 
\end{cases} \]

where \( a \) is some constant.

1. Compute
   
   (a) \( \lim_{x \to 0^+} f(x) \),  
   (b) \( \lim_{x \to 0^-} f(x) \),  
   (c) \( f(0) \)

2. Determine for what value of \( a \) is \( f(x) \):
   
   (d) continuous at \( x = 0 \) from the left
   
   (e) continuous at \( x = 0 \) from the right
   
   (f) continuous at \( x = 0 \)

Solutions:

a) \( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{a}{x} - \frac{2a}{x^2 + x} = \lim_{x \to 0^+} \frac{a(x+2) - 2a}{x^2 + x} \)

   \[ = \lim_{x \to 0^+} \frac{ax + 2a - 2a}{x(x+2)} = \lim_{x \to 0^+} \frac{ax}{x(x+2)} = \lim_{x \to 0^+} \frac{a}{x+2} \]

   \[ = \frac{a}{2} \]

b) \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x^2}{\sqrt{2ax^2+1} - \sqrt{ax^2+1}} \). Rationalizing, we get

   \[ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x^2}{\sqrt{2ax^2+1} - \sqrt{ax^2+1}} \cdot \frac{\sqrt{2ax^2+1} + \sqrt{ax^2+1}}{\sqrt{2ax^2+1} + \sqrt{ax^2+1}} \]

   \[ = \lim_{x \to 0^-} \frac{x^2 (\sqrt{2ax^2+1} + \sqrt{ax^2+1})}{(2ax^2+1) - (ax^2+1)} = \lim_{x \to 0^-} \frac{x^2 (\sqrt{2ax^2+1} + \sqrt{ax^2+1})}{ax^2} \]

   \[ = \lim_{x \to 0^-} \frac{\sqrt{2ax^2+1} + \sqrt{ax^2+1}}{a} = \frac{2}{a} \]

c) \( f(0) = 1 \)
d) In order for $f(x)$ to be continuous at $x = 0$ from the left, we need $\lim_{x \to 0^-} f(x) = f(1)$. Using the computations above, $\frac{2}{a} = 1$, hence $a = 2$.

e) In order for $f(x)$ to be continuous at $x = 0$ from the right, we need $\lim_{x \to 0^+} f(x) = f(1)$. Using the computations above, $\frac{a}{2} = 1$, hence $a = 2$.

f) If $f(x)$ is continuous at $x = 0$, then it is both continuous from the right AND continuous from the left; therefore $a$ needs to satisfy both part (d) and (e). the only possibility is $a = 2$. 
**Problem:** Consider the function
\[ f(x) = \begin{cases} \frac{x^2 + x - 2}{\sqrt{2ax - 2a + 1} - \sqrt{ax - a + 1}} & x < 1 \\ \frac{1}{x} & x = 1 \\ \frac{ax^2 + ax - 2a}{x - 1} & x > 1 \end{cases} \]
where \( a \) is some constant.

1. Compute
   \[ (a) \lim_{x \to 1^+} f(x), \quad (b) \lim_{x \to 1^-} f(x), \quad (c) f(1) \]

2. Determine for what value of \( a \) is \( f(x) \):
   \[ (d) \text{ continuous at } x = 1 \text{ from the left} \]
   \[ (e) \text{ continuous at } x = 1 \text{ from the right} \]
   \[ (f) \text{ continuous at } x = 1 \]

**Solutions:**

a) \( \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{ax^2 + ax - 2a}{x - 1} = \lim_{x \to 1^+} \frac{a(x^2 + x - 2)}{x - 1} = \frac{a(1)^2 + 1 - 2}{1 - 1} = \frac{a}{1} = 3a \)

b) \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x^2 + x - 2}{\sqrt{2ax - 2a + 1} - \sqrt{ax - a + 1}} \). Rationalizing, we get
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{(x - 1)(x + 2)}{\sqrt{2ax - 2a + 1} - \sqrt{ax - a + 1}} \cdot \frac{\sqrt{2ax - 2a + 1} + \sqrt{ax - a + 1}}{\sqrt{2ax - 2a + 1} + \sqrt{ax - a + 1}}
\]
\[
= \lim_{x \to 1^-} \frac{(x - 1)(x + 2)(\sqrt{2ax - 2a + 1} + \sqrt{ax - a + 1})}{(2ax - 2a + 1) - (ax - a + 1)}
\]
\[
= \lim_{x \to 1^-} \frac{(x - 1)(x + 2)(\sqrt{2ax - 2a + 1} + \sqrt{ax - a + 1})}{ax - 1}
\]
\[
= \lim_{x \to 1^-} \frac{(x + 2)(\sqrt{2ax - 2a + 1} + \sqrt{ax - a + 1})}{a} = \frac{6}{a}
\]

c) \( f(1) = 1 \)
d) In order for $f(x)$ to be continuous at $x = 1$ from the left, we need $\lim_{x \to 0^-} f(x) = f(1)$. Using the computations above, $\frac{6}{a} = 1$, hence $a = 6$.

e) In order for $f(x)$ to be continuous at $x = 1$ from the right, we need $\lim_{x \to 0^+} f(x) = f(1)$. Using the computations above, $3a = 1$, hence $a = \frac{1}{3}$.

f) If $f(x)$ is continuous, then it is both continuous from the right AND continuous from the left; therefore $a$ needs to satisfy both part (d) and (e). But there are no values of $a$ that satisfy both condition at the same time, so $f(x)$ can never be continuous.
QUIZ 2

Wed Feb 2, 8-9am

Problem: Consider the function

\[ f(x) = \begin{cases} \frac{(a+x)^{-1} - a^{-1}}{x} & x < 0 \\ -1 & x = 0 \\ \frac{2\sqrt{2x+1} - \sqrt{x+1}}{ax} & x > 0 \end{cases} \]

where \( a \) is some constant.

1. Compute 
   
   \( (a) \lim_{x \to 0^+} f(x), \quad (b) \lim_{x \to 0^-} f(x), \quad (c) f(0) \)

2. Determine for what value of \( a \) is \( f(x) \):
   
   \( (d) \) continuous at \( x = 0 \) from the left
   
   \( (e) \) continuous at \( x = 0 \) from the right
   
   \( (f) \) continuous at \( x = 0 \)

Solutions:

a) \( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2\frac{\sqrt{2x+1} - \sqrt{x+1}}{ax}. \) Rationalizing, we get

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2\frac{\sqrt{2x+1} - \sqrt{x+1}}{ax} \cdot \frac{\sqrt{2x+1} + \sqrt{x+1}}{\sqrt{2x+1} + \sqrt{x+1}} = \lim_{x \to 0^+} 2\frac{x}{ax(\sqrt{2x+1} + \sqrt{x+1})} = \lim_{x \to 0^+} \frac{1}{a(\sqrt{2x+1} + \sqrt{x+1})} = \frac{1}{a}
\]

b) \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{(a+x)^{-1} - a^{-1}}{x} \)

\[
= \lim_{x \to 0^-} \frac{-1 - \frac{1}{a}}{x} = \lim_{x \to 0^-} \frac{a-(a+x)}{x(a+x)} = \lim_{x \to 0^-} \frac{-x}{a(a+x)} \cdot \frac{1}{x} = \lim_{x \to 0^-} \frac{-1}{a(a+x)} = -\frac{1}{a^2}
\]
c) \( f(0) = -1 \)

d) In order for \( f(x) \) to be continuous at \( x = 0 \) from the left, we need \( \lim_{x \to 0^-} f(x) = f(1) \). Using the computations above, \( \frac{-1}{a^2} = -1 \), hence \( a = \pm 1 \).

e) In order for \( f(x) \) to be continuous at \( x = 0 \) from the right, we need \( \lim_{x \to 0^+} f(x) = f(1) \). Using the computations above, \( \frac{1}{a} = -1 \), hence \( a = -1 \).

f) If \( f(x) \) is continuous at \( x = 0 \), then it is both continuous from the right \textbf{AND} continuous from the left; therefore \( a \) needs to satisfy both part (d) and (e). The only possibility is \( a = -1 \).
**QUIZ 2**

Wed Feb 2, 9-10am

**Problem:** Consider the function

\[
f(x) = \begin{cases} 
  \frac{(a+x)^{-1}-a^{-1}}{x} & x < 0 \\
  -1 & x = 0 \\
  \frac{2\sqrt{x+1}-\sqrt{x+1}}{ax} & x > 0 
\end{cases}
\]

where \(a\) is some constant.

1. Compute
   
   (a) \( \lim_{x \to 0^+} f(x) \),  
   (b) \( \lim_{x \to 0^-} f(x) \),  
   (c) \( f(0) \)

2. Determine for what value of \(a\) is \(f(x)\):
   
   (d) continuous at \(x = 0\) from the left
   
   (e) continuous at \(x = 0\) from the right
   
   (f) continuous at \(x = 0\)

**Solutions:**

a) \( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2\frac{\sqrt{2x+1} - \sqrt{x+1}}{ax} \). Rationalizing, we get

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2\frac{\sqrt{2x+1} - \sqrt{x+1}}{ax} \cdot \frac{\sqrt{2x+1} + \sqrt{x+1}}{\sqrt{2x+1} + \sqrt{x+1}} 
\]

\[
= \lim_{x \to 0^+} 2\frac{(2x+1) - (x+1)}{ax(\sqrt{2x+1} + \sqrt{x+1})} = \lim_{x \to 0^+} 2\frac{x}{ax(\sqrt{2x+1} + \sqrt{x+1})} 
\]

\[
= \lim_{x \to 0^+} \frac{1}{a(\sqrt{2x+1} + \sqrt{x+1})} = \frac{1}{a} 
\]

b) \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{(a+x)^{-1} - a^{-1}}{x} \)

\[
= \lim_{x \to 0^-} \frac{1}{a+x} - \frac{1}{a} = \lim_{x \to 0^-} \frac{a-(a+x)}{x(a+x)} \cdot \frac{x}{x} 
\]

\[
= \lim_{x \to 0^-} \frac{a - (a+x)}{a(a+x)} \cdot \frac{x}{x} = \lim_{x \to 0^-} \frac{a-x}{a(a+x)} = \frac{1}{a^2} 
\]
c) $f(0) = -1$

d) In order for $f(x)$ to be continuous at $x = 0$ from the left, we need $\lim_{x \to 0^-} f(x) = f(1)$. Using the computations above, $-\frac{1}{a^2} = -1$, hence $a = \pm 1$.

e) In order for $f(x)$ to be continuous at $x = 0$ from the right, we need $\lim_{x \to 0^+} f(x) = f(1)$. Using the computations above, $\frac{1}{a} = -1$, hence $a = -1$.

f) If $f(x)$ is continuous at $x = 0$, then it is both continuous from the right AND continuous from the left; therefore $a$ needs to satisfy both part (d) and (e). The only possibility is $a = -1$. 