Problem: Given a cylinder with total surface area $A = 6\pi$, find its maximal possible volume.

Solution. The function we have to maximize, is $V = \pi r^2 h$ where $h$ is the height of the cylinder, and $r$ is the base radius. Since there are 2 variables ($r$, and $h$) we need to express one in terms of the other.

A cylinder of height $h$ and radius of the base $r$, has surface area

$$A = A_{\text{bases}} + A_{\text{side}} = 2\pi r^2 + 2\pi hr = 2\pi r(r + h).$$

So we know in our case $6\pi = 2\pi r(r + h)$, that is $r(r + h) = 3$, and $h = \frac{3}{r} - r$. We can substitute this $h$ in the volume formula, and obtain

$$V = V(r) = \pi r^2 \left(\frac{3}{r} - r\right) = 3\pi r - \pi r^3, \quad r \geq 0.$$

To find the maximum, first we look at $V'(r) = 0$:

$$3\pi = 3\pi r^2 \quad \Rightarrow \quad r^2 = 1 \quad \Rightarrow \quad r = 1.$$

So $r = 1$ is a critical value, and since $V''(r) = -6\pi r < 0$ then $r = 1$ is the maximum. The corresponding value of the volume is

$$V(1) = 2\pi.$$
**Problem:** Find the maximal area of a rectangle with diagonal of length $D = 4$.

**Solution.** Call $x$ and $y$, respectively, the base and height of the rectangle. We want to maximize the function $A = xy$, and we know that $x^2 + y^2 = 16$. Using this last equation, we find $y = \sqrt{16 - x^2}$ and

$$A(x) = x\sqrt{16 - x^2},$$

that is the function we have to maximize. Notice that $x$ can vary between $x = 0$ (the rectangle collapses to a vertical line) and $x = 4$ (the rectangle collapses to a horizontal line), and in both extremes the area is 0. Let’s find critical points for $A$:

$$0 = A'(x) = \sqrt{16 - x^2} - x \cdot \frac{x}{\sqrt{16 - x^2}} \Rightarrow \frac{x^2}{\sqrt{16 - x^2}} = \sqrt{16 - x^2} \Rightarrow x^2 = 16 - x^2$$

and so $x = \sqrt{8}$ is the only critical value (in the region $[0,4]$ we’re interested in). Moreover, $A(\sqrt{8}) = 8 > 8$ so it is the maximum.
Problem: Among all isosceles triangles with the two equal edges having length $L = 3$, find the maximal area.

Solution. Call $2x$ and $y$, respectively, the base and height of the triangle. We want to maximize the function $A = \frac{1}{2}(2xy) = xy$, and we know that $x^2 + y^2 = 9$ (by using Pythagoras’ theorem on the half-triangles). Using this last equation, we find $y = \sqrt{9 - x^2}$ and

$$A(x) = x\sqrt{9 - x^2},$$

that is the function we have to maximize, for $x$ between 0 (the triangle collapses to a vertical line) and 3 (the triangle collapses to a horizontal line). As the hint suggests, we can just maximize $A^2(x) = x^2(9 - x^2) = 9x^2 - x^4$. Let’s find the critical points of $A^2$:

$$0 = (A^2)' = 18x - 4x^3 = 2x(9 - 2x^2).$$
	hence the possible values are $x = 0$ and $x = \sqrt{\frac{9}{2}}$ (notice that $x = -\sqrt{\frac{9}{2}}$ is a critical point, but we consider only $r \geq 0$). Notice also $A(0) = 0$, $A(\sqrt{\frac{9}{2}}) = \frac{9}{2}$, so the maximum cannot be 0. Since there are no other critical points in $[0, 3]$, then the maximum of $A$ is attained at $x = \sqrt{\frac{9}{2}}$, and the corresponding area is $A = \frac{9}{2}$. 