Notes for Pizza Seminar on The Probabilistic Method with Applications to Ramsey Theory

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Abstract

The probabilistic method is a powerful tool that has become ubiquitous in combinatorics over the past half-century. The field of Ramsey Theory, in particular, makes use of the probabilistic method quite frequently. In this talk, I’ll introduce some key concepts from Ramsey Theory—namely the Ramsey Numbers and the Van der Waerden Numbers—and use the probabilistic method to find out more about them. The ultimate goal of this talk is to show how the probabilistic method can be used to solve daunting deterministic problems. The content of sections (1) – (5) of this talk is taken from The Probabilistic Method by Noga Alon and Joel H. Spencer. Section (6) is taken from the Ramsey Theory on the Integers by Bruce M. Landman and Aaron Robertson.

1 Definition of Ramsey Numbers

We first define a graph: a graph \((V, E)\) is a set of vertices \(V\) together with a set of edges \(E\), which are subsets of \(V\) consisting of two elements. A complete graph on \(n\) vertices, \(K_n\), is a graph \((V, E)\) so that for any distinct \(v, w \in V\), \(\{v, w\} \in E\); this simply means that all possible edges are present.

For positive integers \(k\) and \(\ell\), the Ramsey Number \(R(k, \ell)\) is the smallest integer \(n\) such that in any two-coloring of the edges of \(K_n\) by red and blue, either there is a red \(K_k\) or a blue \(K_\ell\) appearing as a subgraph. From Ramsey himself, we have

**Theorem.** For all \(k\) and \(\ell\),

\[
R(k, \ell) < \infty.
\]

Some basic facts: \(R(k, \ell) = R(\ell, k)\), \(R(2, k) = k\) and \(R(1, k) = 1\). We can (sometimes) calculate them precisely: a basic result of Ramsey Theory is that \(R(3, 3) = 6\). This isn’t too difficult to see, and is a good exercise.
2 An Easy Probabilistic Lower Bound on $R(k, k)$

**Proposition.** If $n$ and $k$ be integers such that
\[
\binom{n}{k} \cdot 2^{1 - \left(\frac{k}{2}\right)} < 1,
\]
then $R(k, k) > n$.

**Proof.** Let’s do the following: randomly color each edge red or blue, independently and uniformly. If the probability that there is no monochromatic $K_k$ is positive (i.e. non-zero), then there must be a coloring that of $K_n$ that doesn’t yield a monochromatic $K_k$. This is the crux of the probabilistic method. To prove this theorem, let’s compute the probability that there is no monochromatic $K_k$.

For each set of $k$ vertices, $S$, let $A_S$ be the probability that $S$ forms a monochromatic $K_k$. We note that for any $S$, $P(A_S) = \left(\frac{1}{2}\right)^{\binom{k}{2}} + \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{1 - \left(\frac{k}{2}\right)}$. We then have
\[
P(\text{There is a monochromatic } K_k) = P\left(\bigcup_S A_S\right) \leq \sum_S P(A_S) = \binom{n}{k} \cdot 2^{1 - \left(\frac{k}{2}\right)} < 1.
\]
This means that
\[
P(\text{There is no monochromatic } K_k) > 0,
\]
i.e. at least one such coloring exists.

A (messy) calculation gives a lower bound:

**Corollary.** $R(k, k) > \frac{1}{\sqrt{2}} (1 + o(1)) k^{2k/2}$. 

We can easily generalize the above result, with a small, almost trivial observation: if $X : \Omega \to \mathbb{R}$ is a random variable with $E[X] < \infty$, then there exist $\omega_1, \omega_2 \in \Omega$ so that $X(\omega_1) \leq E[X]$ and $X(\omega_2) \geq E[X]$. In words, this means that if $X$ is a random variable with finite mean, then there exists an outcome in which $X$ is greater than or equal to its mean, and an outcome in which $X$ is less than or equal to its mean. Recall that if $A$ is an event, then $1_A$ is the indicator random variable for $A$. This implies that $E 1_A = P(A)$.

We claim that for any $a$ and $n$, there is a two-coloring of $K_n$ with at most
\[
\binom{n}{a} \cdot 2^{1 - \left(\frac{a}{2}\right)}
\]
monochromatic $K_a$. This is almost identical to the proof above.

3 A Minor Alteration Improves Our Bound

We can alter our proof slightly to get a better bound.
**Proposition.** For any $n$ and $k$, we have

$$R(k, k) > n - \binom{n}{k} 2^{1 - \frac{k}{2}}.$$ 

**Proof.** Randomly two-color each edge of $K_n$ independently and uniformly. We know from the above that there is a coloring such that with at most $\binom{n}{k} 2^{1 - \frac{k}{2}}$ monochromatic $K_k$'s. Remove one vertex from each such graph; we have removed at most $\binom{n}{k} 2^{1 - \frac{k}{2}}$ such vertices, and there is no monochromatic $K_k$ on the remaining graph on at least $n - \binom{n}{k} 2^{1 - \frac{k}{2}}$ vertices. 

Some calculation—i.e. taking $n \sim e^{-1} k^{2/k} (1 - o(1))$ gives a lower bound:

**Corollary.** For any $k$,

$$R(k, k) > \frac{1}{e} (1 + o(1)) k^{2/k}.$$ 

This is a (very small) improvement over our previous bound.

## 4 The Lovász Local Lemma Improves It Further

Let us first state the local lemma:

**Theorem. The Symmetric Local Lemma:** Let $A_1, A_2, \ldots, A_n$ be events. Suppose that each $A_i$ is mutually independent of a set of all the other events $A_j$ but at most $d$, and that $P[A_i] \leq p$ for all $1 \leq i \leq n$. If

$$ep(d + 1) \leq 1$$ 

then

$$P[\bigwedge_{i=1}^n \overline{A_i}] > 0.$$ 

The Local Lemma can help us bound $R(k, k)$ further:

**Proposition.** If $e \binom{k}{2} \binom{n-2}{k-2} \cdot 2^{1 - \frac{k}{2}} < 1$ then $R(k, k) > n$.

**Proof.** Randomly two-color each edge, independently and uniformly. For each $k$–set of vertices $S$, define $A_S$ to be the event that the complete graph on $S$ is monochromatic. Then we have that $P(A_S) = 2^{1 - \binom{k}{2}}$. For any two $k$–sets of vertices $S$ and $T$, we have that $A_S$ and $A_T$ are independent unless $|S \cap T| \geq 2$. This implies that each event $A_S$ fails to be independent with at most $d < \binom{k}{2} \binom{n-2}{k-2}$ events. Applying the Local Lemma with $p = 2^{1 - \frac{k}{2}}$ gives the desired result. 

More computation gives a (slightly improved) lower bound:

**Corollary.** For any $k$,

$$R(k, k) > \frac{\sqrt{2}}{e} (1 + o(1)) k^{2/k}.$$ 

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Somewhat surprisingly, the above is the best known bound for diagonal Ramsey numbers, and all known proofs of it use the Local Lemma somehow. There is a (more subtle) Local Lemma for less symmetric cases. Using it, you can prove:

**Proposition.**

\[ R(k, 3) > \Omega\left(\frac{k^2}{(\log k)^2}\right) \]

and

**Proposition.**

\[ R(k, 4) > \Omega\left(k^{5/2+o(1)}\right). \]

The bound on \( R(k, 3) \) was improved by a factor of \( \log k \) in 1995, but the bound on \( R(k, 4) \) is the best known bound (and all known proofs use the Local Lemma). These bounds aren’t necessarily hard to derive, per se, but the amount of calculation required is obscene. As far as upper bounds, the best known for the diagonal Ramsey numbers is

**Proposition.**

\[ R(k, k) < (4 + o(1))^k \]

which does not utilize probabilistic methods.

## 5 Multicolored Sets of Real Numbers

For a \( k \)-coloring \( c : \mathbb{R} \to [k] \) of \( \mathbb{R} \), and for a subset \( T \subset \mathbb{R} \), we say that \( T \) is *multicolored* if \( c(T) = [k] \), i.e. if \( T \) contains elements of all colors. We can prove the following:

**Theorem.** Let \( m \) and \( K \) be two positive integers satisfying

\[ e(m(m-1)+1)k \left(1 - \frac{1}{K}\right)^m \leq 1. \]

Then, for any set \( S \) of \( m \) real numbers, there is a \( k \)-coloring so that each translation \( x+S \) (for \( x \in \mathbb{R} \)) is multicolored.

We note that this holds whenever \( m > (3 + o(1))k\log k \), so for any given \( k \), there is an \( m \) large enough.

**Proof.** We first prove that the above holds if we only require translations by a finite subset. Fix \( X \subset \mathbb{R} \), with \( |X| < \infty \). Let \( Y \) be the union of all \( X \) translates of \( S \), that is

\[ Y = \bigcup_{x \in X} (x + S) \]

and let \( c : Y \to [k] \) be a random \( k \)-coloring, with each \( c(y) \) chosen independently and uniformly. For each \( x \in X \), define \( A_x \) to be the event that \( x+S \) is not multicolored. Then for each \( x \), we have \( \mathbb{P}(A_x) \leq k(1 - 1/k)^m \).
Additionally, each $A_x$ fails to be dependent only for events $A_{x'}$ where $(x + S) \cap (x' + S) \neq \emptyset$. There are at most $m(m-1)$ such events, so the local lemma proves that

$$P\left( \bigcap_{x \in X} A_x \right) > 0,$$

i.e. that such a coloring exists.

We now may use a compactness argument to complete the proof. Since $[k]$ is compact under the discrete topology, $[k]^\mathbb{R}$, the space of all functions from $\mathbb{R} \to [k]$, is compact by Tychonoff’s Theorem. For any $x \in \mathbb{R}$, the set $C_x$ of all colorings $c$, such that $x + S$ is multicolored, is closed, and is thus compact; to see that it is closed, note that a basis for the open sets is the set of all colorings whose values are specified on a finite number of points. For any finite set $X$, we know that

$$\bigcap_{x \in X} C_x \neq \emptyset$$

by the above argument. Thus, by the finite intersection property of compact sets, we have

$$\bigcap_{x \in \mathbb{R}} C_x \neq \emptyset.$$

Any coloring from this non-empty intersection has the desired property. \qed

6 Van der Waerden Numbers

For positive integers $r$ and $k$, define $w(k; r)$ to be the smallest $N$ so that every $r$-coloring of $[N]$ has a monochromatic arithmetic progression of $k$ elements. By Van der Waerden himself,

**Theorem.** For any positive integers $r$ and $k$, we have

$$w(k; r) < \infty.$$

We can use the probabilistic method to get a lower bound on the van der Waeden numbers $w(k; 2)$. We need a small lemma first:

**Lemma.** For any positive integers $k$ and $n$, define $A_k(n)$ to be the number of arithmetic progressions of $k$ elements of $[n]$. Then

$$A_k(n) \leq \frac{n^2}{k - 2}.$$

**Proof.** We simply count them by step size: if $a$ is a step size so that there is at least one arithmetic progression of $k$ elements of $[n]$, then we have $n - a(k - 1)$ such progressions. Since $a \leq \frac{k-1}{k-1}$, we have

$$A_k(n) \leq \sum_{a=1}^{\left\lfloor \frac{(n-1)(k-1)}{(k-1)} \right\rfloor} n - a(k - 1) \leq \frac{n^2}{k - 2}.$$

\qed
This easily gives a lower bound:

**Proposition.** If

\[ \frac{n^2}{(k - 1)2^k} < 1, \]

then

\[ w(k; 2) > n. \]

**Proof.** Color each integer in \([n]\) randomly. Then the probability that any given arithmetic progression is monochromatic is \(2^{1-k}\). Since there are at most \(A_k(n)\) progressions, the lemma implies that the probability there is an arithmetic progression is < 1. This means that there exists a coloring so that there is no arithmetic progression.

A quick calculation once again gives

**Corollary.**

\[ w(k; 2) > \sqrt{k - 1} \cdot 2^{k/2 - 1}. \]