Notes for Graduate Student Seminar on Random Graphs and Their
Zero-One Law
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Abstract

Random graphs are a fundamental object of probabilistic combinatorics, and have far-reaching applications in engineering, computer science, and applied mathematics. We’ll focus just on their mathematical properties and cover two of their zero-one laws which provide a surprising intersection between probability and mathematical logic. No knowledge of probability, graph theory, or logic will be assumed.

1 What Is A Random Graph?

Before defining a random graph, let’s first discuss what a graph is. A (simple, undirected) graph $G$ is a pair $(V,E)$ where $V$ is a set whose elements are called vertices and $E$ is a set of unordered pairs of vertices. Given $u,v \in V$, we say $u \sim v$ if $\{u,v\} \in E$.

Definition. The Erdős Rényi random graph $G(n,p)$ is a graph whose vertex set is $\{1,2,\ldots,n\}$ and whose edge set satisfies $\mathbb{P}\{i,j \in E\} = p$ for each $1 \leq i < j \leq n$ and all of these events are independent of each other.

In words, $G(n,p)$ is the graph on $n$ vertices with each possible edge is independently added with probability $p$. Random graphs were first introduced by Paul Erdős and Alfred Rényi in seminal their 1960 paper On the Evolution of Random Graphs, in which they mainly asked about the size of the largest connected component of $G(n,p)$. Ordinarily, the questions involve taking $n \to \infty$ and allowing $p$ to depend on $n$.

While this talk is focused on their mathematical theory, random graphs are widely used to model social networks, wireless networks, and various other applications related to connectivity.

2 Two Easy Properties

If we fix $p \in (0,1)$ and take $n$ quite large, is it likely that $G(n,p)$ contains an isolated vertex? Intuitively, the answer should be no, and indeed the answer is no. For a fixed $n$, define $X$ to be the number of vertices in
$G(n, p)$ with no neighbors. We can break $X$ up into a sum of indicator variables: $X = X_1 + \cdots + X_n$ where

$$X_i = \begin{cases} 
1 & \text{if } i \text{ has no neighbors} \\
0 & \text{otherwise}
\end{cases}$$

Since $X$ is non-negative integer-valued, $P(X > 0) = P(X \geq 1) \leq EX$. This gives us the bound:

$$P(X > 0) \leq EX = \sum_i EX_i = nEX_1 = n(1 - p)^{n-1}.$$ 

For any fixed $p \in (0, 1)$, this value goes to zero as $n \to \infty$. Thus, we see:

**Proposition.** For $p \in (0, 1)$, $P(G(n, p) \text{ has an isolated vertex}) \to 0$.

This method—where the mean (which is also known as the first moment) going to zero shows a probability goes to zero—is known as the first moment method. It’s very good for showing that something doesn’t happen. What if we want to show that something does happen? Let’s look at an example.

For a fixed $p \in (0, 1)$ and $n$ large, it seems quite likely that $G(n, p)$ contains a triangle. If we call $X$ the number of triangles, this is asking for a lower bound on $P[X > 0]$. The first moment method tells us that $P[X > 0] \leq EX$, but how can we get a lower bound? The answer is an inequality usually referred to as the second moment method:

**Theorem.** (Paley-Zygmund Inequality) If $X$ is a non-negative random variable, then

$$P[X > 0] \geq \frac{(EX)^2}{E[X^2]}.$$ 

**Proof.** Cauchy-Schwarz inequality.

In order to show that $P[X > 0] \to 1$, it is thus sufficient to show that $\frac{(EX)^2}{E[X^2]} \to 1$. Let’s start with the first moment: there are $\binom{n}{3}$ unordered triples. Fix an ordered of these triples; we may similarly write $X = X_1 + \cdots + X_{\binom{n}{3}}$ where $X_i$ is 1 if there is a triangle on the $i$th triple, and 0 otherwise. We can then calculate:

$$EX = \sum_{i=1}^{\binom{n}{3}} EX_i = \binom{n}{3}EX_i = \binom{n}{3}p^3 \sim \frac{n^3}{6}p^3$$

where we write $l_n \sim r_n$ if $\frac{l_n}{r_n} \to 1$ as $n \to \infty$. Now let’s look at the second moment: we have

$$E[X^2] = E \left[ \sum_{i=1}^{\binom{n}{3}} X_i \right]^2 = \sum_i E[X_i^2] + \sum_{i \neq j} E[X_iX_j] = \sum_i E[X_i] + \sum_{i \neq j} E[X_iX_j]$$

The first sum is simply $EX$, while the latter is a bit more complicated. Note that $E[X_iX_j]$ is the probability that there is a triangle at locations $i$ and $j$. If the two triples share 0 or 1 point, then these events are independent, and we have $E[X_iX_j] = p^6$. If the two triples share 2 points, then there are only 5 edges required implying that in this case we have $E[X_iX_j] = p^5$. All we have to do now is count how many such
pairs of triples there are. In the latter case, we have \( \binom{n}{3} \) choices for triple \( i \), \( \binom{3}{2} \) choices for which vertices to share with \( j \), and \( n - 4 \) for the other vertex for \( j \) to have. Thus there are \( \binom{n}{3} \cdot \binom{3}{2} \cdot (n - 4) \sim \frac{n^4}{3} \) arrangements in the second case. Since the total number of ordered pairs of triples is \( \binom{n}{3} \cdot \binom{n}{3} - 1 \sim \frac{n^6}{36} \), we must have that the number of arrangements in the first case is \( \sim \frac{n^6}{36} - \frac{n^4}{2} \sim \frac{n^6}{36} \). Putting this all together gives

\[
\frac{(\mathbb{E}X)^2}{\mathbb{E}X^2} \sim \frac{(n^3/6)^2 p^6}{(n^3/6)p^3 + (n^6/36)p^6 + (n^4/6)p^5} \rightarrow 1.
\]

This proves

**Proposition.** For a fixed \( p \in (0, 1) \), \( \mathbb{P}[G(n, p) \text{ has a triangle}] \rightarrow 1 \).

You could similarly prove this proposition by breaking up \( \{1, 2, \ldots, n\} \) into \( n/3 \) disjoint triples, and using independence to get an upper bound on the probability that there is no triangle. More precisely, for \( i = 1, 2, \ldots, \lfloor n/3 \rfloor \), let \( A_i \) bet he event that configuration \( i \) has no triangle. Then we have

\[
\mathbb{P}[G(n, p) \text{ has no triangles}] \leq \mathbb{P}\left[ \bigcap_{i=1}^{\lfloor n/3 \rfloor} A_i \right] = \prod_{i=1}^{\lfloor n/3 \rfloor} \mathbb{P}[A_i] = (1 - p^3)^{\lfloor n/3 \rfloor}.
\]

Taking \( n \rightarrow \infty \) shows that this probability goes to zero.

### 3 A First Zero-One Law

The above argument can be used to show that for any fixed finite graph, \( H \), we have

\[
\mathbb{P}[H \text{ is a subgraph of } G(n, p)] \rightarrow 1
\]

for any fixed \( p \in (0, 1) \). It turns out that having a certain subgraph is one of many properties in a large family of properties, called first order properties.

**Definition.** A first-order graph property is one that can be expressed using variables representing vertices of a graph, equality, adjacency (denoted \( x = y \) and \( x \sim y \) respectively), Boolean connectives (e.g. \( \land, \lor, \neg \)), and universal and existential quantification (\( \forall x, \exists y \)). Finally, each sentence must be finite.

For example, the property of containing a triangle can be written as

\[
\exists x \exists y \exists z [x \sim y \land x \sim z \land y \sim z].
\]

Similarly, the property of having no isolated point can be written as

\[
\forall x \exists y [x \sim y].
\]

Somewhat shockingly, every first order property either happens always or never asymptotically:

**Theorem.** (Glebskii, Kogan, Liagonkii, Talanov, 1969) For a fixed \( p \in (0, 1) \) and first order property \( A \),

\[
\lim_{n \to \infty} \mathbb{P}[G(n, p) \text{ satisfies } A] \in \{0, 1\}.
\]
Proving this is difficult, and beyond the scope of this talk. The easiest proof is from model theory, a subject from mathematical logic. The basic idea is that you can take a limit of your finite graphs \( G(n, p) \) to get a countable graph known as the Rado graph (sometimes called the random graph). This notion of taking such a limit of structures is called a Fraïssé Limit; independent of \( p \), the limiting object you get from \( G(n, p) \) is a.s. the same object: the Rado graph. The first-order property effectively assures some sort of continuity, in that the limit of probabilities is the probability of the limit. Another way to prove the zero-one law is through the use of the Ehrenfeucht-Fraïssé game (EF game). The EF game is a full-information game that translates the first-order property into a winning condition for the game.

Let’s go through some more examples of first-order statements: for instance, the property that a graph has diameter at most two can be written as

\[ \forall x \forall y [x = y \lor x \sim y \lor (\exists z [x \sim z \land z \sim y])]. \]

Since this is first-order, the probability that \( G(n, p) \) has diameter at most two is either asymptotically true or false, and it’s not too difficult to convince yourself that it’s true.

A benefit of the zero-one is that it can be used to prove that certain properties are not first-order. In fact, it’s quite difficult to prove such a statement without extra machinery, but we can use (the converse to) the zero-one law. Here’s an example:

**Proposition.** The property that finite graph has an even number of vertices is not first-order.

**Proof.** This follows by noting that

\[ \mathbb{P}[G(n, p) \text{ has an even number of vertices}] = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}. \]

Thus there is no limit as \( n \to \infty \). By (the contrapositive of) the zero-one law, this is not a first order property. \( \square \)

This is an instance of the zero-one law failing in that the limit doesn’t exist. An example of the limit existing but not being equal to zero or one, consider the following:

**Proposition.** The property that a finite graph has an even number of edges is not first-order. In particular:

\[ \mathbb{P}[G(n, 1/2) \text{ has an even number of edges}] = 1/2 \text{ for each } n. \]

**Proof.** Note that \( G(n, 1/2) \) chooses a (labeled) graph uniformly at randomly. Thus, to show the above, it is sufficient to show that the number of graphs on \( n \) vertices with an even number of edges—call this set \( E_n \)—is equal to the number with an odd number of edges, which we call \( O_n \). We can define a bijection from \( E_n \) to \( O_n \) as follows: given a graph with an even number of edges, if it contains the edge \( \{1, 2\} \), remove it. If it doesn’t contain it, add it. This is clearly a bijection between the two sets, implying that \( \#E_n = \#O_n \). \( \square \)
In general, it is difficult to come up with examples of inherently probabilistic properties where we have a limit other than 0 or 1 for when we fix a probability. In order to get more interesting behavior, we must begin to vary the probability with the number of nodes.

4 Threshold Functions

Let's return to our first example, in which we ask whether or not \( G(n, p) \) contains a triangle. We saw that for a fixed probability, we are guaranteed to have a triangle as we take \( n \to \infty \). If we now allow the probability to vary, i.e. if we write \( p(n) = p(n) \), how fast can \( p(n) \to 0 \) and still guarantee that we have a triangle? More formally, we are asking for a threshold function:

**Definition.** Given a property \( A \) of graphs, a function \( f : \mathbb{N} \to [0, 1] \) is a threshold function if

- If \( p(n) \gg f(n) \) then \( \mathbb{P}[G(n, p) \text{ satisfies } A] \to 1 \),
- If \( p(n) \ll f(n) \) then \( \mathbb{P}[G(n, p) \text{ satisfies } A] \to 0 \)

or vice versa.

Here, \( p(n) \gg f(n) \) means \( p(n)/f(n) \to \infty \) and \( p(n) \ll f(n) \) means \( p(n)/f(n) \to 0 \). In a sense, a threshold function tells us the critical rate at which the property occurs: if the probability is much larger or much smaller than the threshold, the property either always or never holds. Note that a threshold function isn’t unique: for instance, \( 1/n \) and \( 1/n \) and \((n + \log(n))/n^2 \) are asymptotically equal, so they are equivalent as thresholds. Note also that we can’t say anything *a priori* about the behavior of \( G(n, p) \) at a threshold.

What, then, is a threshold function for having a triangle?

**Proposition.** A threshold function for containing a triangle is \( 1/n \).

**Proof.** We need to show that if \( p(n) \gg 1/n \) then we always asymptotically almost surely have a triangle, and if \( p(n) \ll 1/n \), then we do not. The latter is easier in this case. Assume \( p(n) \ll 1/n \). Then, by the first moment method,

\[
\mathbb{P}[G(n, p(n)) \text{ has a triangle}] \leq \binom{n}{3} p^3 \sim \frac{n^3}{6} p^3 \to 0
\]

since \( p \ll 1/n \) implies \( np \to 0 \). Now let’s examine the reverse: suppose \( p(n) \gg 1/n \). As usual, this requires a lower bound on the probability that we have a triangle, and the second moment method is the best way to press forward. From before, we have

\[
\mathbb{P}[G(n, p) \text{ has a triangle}] \geq \frac{(\mathbb{E}X)^2}{\mathbb{E}[X^2]}
\]

\[
\sim \frac{(n^3/6)^2 p^6}{(n^3/6)p^3 + (n^6/36)p^6 + (n^4/6)p^5}
\]

\[
= \frac{(np)^6/36}{(np)^3/6 + (np)^6/36 + (np)^4p/6}
\]

\[
\to 1
\]
since $np \to \infty$.

Quite often, the most complex behavior occurs when $p$ is on the same order as the threshold function. In the case of triangles, we have the following surprising result:

**Proposition.** Let $p = c/n$ for $c > 0$, and define $X$ to be the number of triangles contained in $G(n, p)$. Then as $n \to \infty$, $X$ converges in distribution to a Poisson random variable with mean $c^3/6$.

**Proof.** The easiest way of proving this is by the method of (factorial) moments, i.e. to show that $E(X^r) \to (c^3/6)^r/r!$ for each fixed $r$.

Can we generalize the above to other finite graphs? That is, for any given finite graph $H$, can we easily find its threshold function? A naive guess would be that if $H$ has $v$ vertices and $e$ edges then the threshold function for $H$ is $n^{-v/e}$, but this isn’t true unfortunately. To see a counterexample, consider a large complete graph with a single edge added from an existing vertex to a new vertex. The ratio $e/v$ is some notion of density, and in the case of a complete graph with an extra edge added, the density of it isn’t captured just by counting the edges and dividing by the number of vertices since the extra edges messes that up. It turns out that it depends on the maximal density.

**Theorem.** (Erdős, Rényi, 1960) Let $H$ be a finite graph and $H_1$ the subgraph of $H$ with maximal density $e_1/v_1$. Then $n^{-e_1/v_1}$ is a threshold function for the property that $G(n, p)$ contains a copy of $H$.

**Proof.** Careful use of first and second moments.

Since we note that each such threshold function for a subgraph is of the form $n^{-q}$ for some $q \in \mathbb{Q}$, it’s natural to ask whether or not there are properties of the form $n^{-\alpha}$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Perhaps surprisingly, it leads us to another zero-one law about first-order logic:

**Theorem.** (Shelah and Spencer, 1988) For any irrational $\alpha \in (0, 1)$ and any first order $A$, we have

$$\lim_{n \to \infty} P[G(n, n^{-\alpha}) \text{ satisfies } A] \in \{0, 1\}.$$ 

Note, that this doesn’t say that the interesting thresholds are all at $n^{-q}$ for $q$ rational, but simply that nothing happens at $n^{-\alpha}$ for $\alpha$ irrational. In the words of Alon and Spencer: “What happens in the evolution of $G(n, p)$ at $p = n^{-\pi/7}$? The answer: Nothing!” It is still possible to have a first-order property with a more complicated threshold function, e.g.:

**Proposition.** A threshold property for having an isolated vertex is $p(n) = \frac{\log(n)}{n}$.

**Proof.** First and second moment methods, see [4].

The proof of this second zero-one law is similar; for each irrational number, we know that for any specific subgraph, the threshold function has rational exponent so it either appears or doesn’t appear in $G(n, n^{-\alpha})$ a.s. This allows us to construct a Fraïssé limit for each $\alpha$ (but this limit depends on $\alpha$), thereby giving us a zero-one law.
5 Largest Connected Component

Before closing, there are a few stray facts worth mentioning about connectivity. It’s not clear off the bat whether or not connectivity is a first-order property, so we can’t use our zero-one law to find its threshold function. It turns out to be:

**Proposition.** The threshold function for connectivity is \( p(n) = \log(n)/n \)

**Proof.** Show that the probability of being disconnected is asymptotically equal to the probability of having an isolated vertex, and use the threshold function for isolated vertex. See [2].

In Erdős and Rényi’s classic paper, they specifically looked at the size of the largest component of \( G(n, p) \) for \( p(n) = \Theta(1/n) \). While they found five distinct regions with wildly different behavior, we only present the three:

**Theorem.** For each \( k \), let \( L_k \) denote the size of the \( k \)th largest connected component. Then for \( G(n, p) \) with \( p = c/n \) for \( c > 0 \), we have

- If \( c < 1 \), then \( L_1 = \Theta(\log n) \) and \( L_k \sim L_1 \) for each fixed \( k \)
- If \( c = 1 \), then for each fixed \( k \), \( L_k = \Theta(n^{2/3}) \)
- If \( c > 1 \), then \( L_1 \sim yn \) where \( y \) is the positive real solution to \( e^{-cy} = 1 - y \). Moreover, \( L_2 = \Theta(\log n) \).

There is also a central limit theorem for \( L_1 \):

\[
\frac{L_1 - yn}{\sqrt{n}} \rightarrow N \left( 0, \frac{y(1-y)}{1-c(1-y)} \right)
\]

**Proof.** See [2] and [3].

6 Automorphisms and Symmetry

Here we discuss the amount of symmetry present in random graphs, which is captured in the automorphism group:

**Definition.** An automorphism of a graph \( G = (V, E) \) is a bijection \( \phi : V \rightarrow V \) s.t. \( v \sim w \iff \phi(v) \sim \phi(w) \). Define \( \text{Aut}(G) \) to be the group of automorphisms of \( G \) under composition.

The automorphism group can be the trivial group or as large as \( S_{|V|} \). It’s difficult to come up with arbitrarily large graphs with no automorphism, but for each \( n \) the complete graph \( K_n \) has \( \text{Aut}(K_n) = S_n \). In some sense, the graph \( K_n \) is the most symmetric graph; we can use the automorphism group to give a precise definition of what a symmetric graph is:
**Definition.** A graph $G$ is symmetric if $|\text{Aut}(G)| > 1$. It is asymmetric if $\text{Aut}(G) = \{\text{id}\}$. Moreover, define the degree of symmetry of a graph to be

$$A(G) = \min\{|E(G) \triangle E(H)| : H \text{ is a symmetric graph on } V\}.$$  

In words, $A(G)$ is the smallest total number of changes (i.e. additions and deletions of edges) that can make $G$ symmetric. Note that if a graph is symmetric, then $A(G) = 0$. This leads us to the following statement:

**Theorem.** Almost surely, $A(G(n, 1/2)) \sim \frac{n^2}{2}$. In particular, $G(n, 1/2)$ is asymmetric almost surely.

**Proof.** See [2].

While the first part of the theorem concerning the degree of asymmetry is quite difficult to prove, the latter can be done elementarily. The easiest proof of this is to use the first moment method, summing over all set theoretic bijections and the grouping them together by the number of fixed points they have and showing this sum goes to zero. This also gives a proof of the following non-trivial deterministic fact:

**Theorem.** For all sufficiently large $n$, there exists an asymmetric graph with $n$ vertices.

**References**


