Some solutions to HW 9

Problem 1) Prove that dot product $\langle \cdot , \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

Fix points $(a_1, b_1) \in \mathbb{R}^n \times \mathbb{R}^n$.

Then if $\delta > 0$ and we have points $(a_2, b_2) \in \mathbb{R}^n \times \mathbb{R}^n$, we have

$$|\langle a_1, b_1\rangle - \langle a_2, b_2\rangle| = |\langle a_1, b_1\rangle - \langle a_1, b_2\rangle + \langle a_1, b_2\rangle - \langle a_2, b_2\rangle| =$$
$$= |\langle a_1, b_1 - b_2\rangle + \langle a_1 - a_2, b_2\rangle| \le \|a_1\| \cdot \|b_1 - b_2\| + \|a_1 - a_2\| \, \|b_2\|$$

By Cauchy-Schwartz and triangle ineq.

Then if $\|a_1 - a_2\| < \delta$ and $\|b_1 - b_2\| < \delta$,

then $|\langle a_1, b_1\rangle - \langle a_2, b_2\rangle| < (\|a_1\| + \|b_2\|)\delta$

So taking δ small enough we ensure that $|\langle a_1, b_1\rangle - \langle a_2, b_2\rangle|$ is small.

Pg 173 #10) Claim: M is connected and locally path-connected $\iff$ M is path-connected.

Proof: "$\Leftarrow$": is easy: path-connected space is connected, and any point has a path-conn. neighborhood which is the whole space.

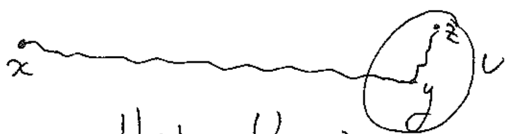
"$\Rightarrow$": Suppose M connected and loc. path-connected.
Pick an arbitrary $x \in M$. We want to prove
$$\forall y \in M \, (\exists \text{ path } x \rightsquigarrow y)$$
Denote $U_x = \{ y \in M \mid \exists \text{ path } x \rightsquigarrow y \}$.
If we can prove the following:

1) U_x open
2) $M \setminus U_x$ open
3) $U_x \neq \emptyset$

Then we can conclude that $U_x = M$, since M is connected.
In that case we could conclude that M is path-connected

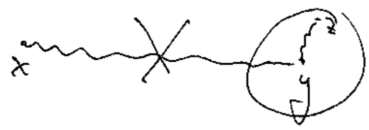
Let's now prove 1), 2) and 3):

1) If $y \in U_x$ and U is a path-connected neighborhood of y, then $U \subseteq U_x$: if $z \in U$, $\exists$ path $y \rightsquigarrow z$ and $\exists$ path $x \rightsquigarrow y$, so $\exists$ path $x \rightsquigarrow z$.



This means that U_x is open.

2) If $y \in M \setminus U_x$ and U is path-conn. nbh of y, then $U \subseteq M \setminus U_x$: if $z \in U$, suppose $z \in U_x$. then $\exists$ path $x \rightsquigarrow z$ and $\exists$ path $z \rightsquigarrow y$, so $\exists$ path $x \rightsquigarrow y$ $\lightning$ contradiction



Thus $z \notin U_x$, so $z \in M \setminus U_x$ and $U \subseteq M \setminus U_x$

So $M \setminus U_x$ is open

3) Clearly $x \in U_x$ ($\exists$ constant path $x \rightsquigarrow x$), so $U_x \neq \emptyset$.

This concludes the proof.

Pg 182 #5) e.g. $f(x) = \sin(x)$, $U = (0, 10)$. Then $f(U) = [-1, 1] \subseteq \mathbb{R}$ not open.

Pg 184 #1) ($f: \mathbb{R} \to \mathbb{R}$ cont)

		closed	open	compact	connected
a)	$f^{-1}(\{0\})$	YES	NO	NO	NO
b)	$f^{-1}((1,\infty))$	NO	YES	NO	NO
c)	$f([0,\infty))$	NO	NO	NO	YES
d)	$f([0,1])$	YES	NO	YES	YES

#3) e.g. $f(x) = \frac{1}{1+x^2}$, $B = \mathbb{R}$. Then $f(B) = (0, 1] \subseteq \mathbb{R}$ not closed.

b) if B bounded and closed, then B compact $\Rightarrow f(B)$ compact $\Rightarrow f(B)$ closed.

#4) We have continuous surjective map $f: A \times B \to A$ def. by $f(a,b) = a$.
Thus $A = f(A \times B)$ is connected.

Pg 191 #1) e.g. $\arctan(x)$, or $\frac{1}{1+x^2}$.

#3) Let $m = \max(f(K))$. Then $M = f^{-1}(\{m\}) \subseteq K$ is closed and bounded, thus compact.

#4) The values that f takes on the curve are exactly
$$\{ f(c(t)) \mid t \in [0,1] \} = (f \circ c)([0,1]).$$
As $f \circ c$ is continuous, $(f \circ c)([0,1])$ is compact, in $\mathbb{R}$ so it achieves the max. and min.

Pg 194 #5) $[0,1]$ is compact, $(0,1)$ is not
$\Rightarrow (0,1)$ is <u>not</u> a continuous image of $[0,1]$.