Some solutions to HW10

Pg 196: #2) $a>0$, $f:[a,\infty)\to\mathbb{R}$ def. by $f(x)=1/x$

Show that f is uniformly continuous:

Note that for all $x,y\in[a,\infty)$ we have

$$|f(x)-f(y)| = \left|\frac{1}{x}-\frac{1}{y}\right| = \frac{|y-x|}{|xy|} \leq \frac{|y-x|}{a^2} = \frac{1}{a^2}|x-y|,$$

so f is $\frac{1}{a^2}$-Lipschitz $\Rightarrow$ uniformly continuous.

#6) From the definition we get equivalence

f not uniformly continuous

$\Leftrightarrow$

$\neg\, \forall \varepsilon>0\ \exists \delta>0:\ \forall x,y\in\mathbb{R}\ \left(|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon\right)$

$\Leftrightarrow$ {Logic}

$\exists \varepsilon>0\ \forall \delta>0:\ \exists x,y\in\mathbb{R}\ \left(|x-y|<\delta \ \wedge\ |f(x)-f(y)|\geq\varepsilon\right)$

$\Leftrightarrow$ {Archimedean property}

$\exists \varepsilon>0\ \forall n\in\mathbb{N}\ \exists x_n,y_n\in\mathbb{R}\ \left(|x_n-y_n|<\frac{1}{n} \ \wedge\ |f(x_n)-f(y_n)|\geq\varepsilon\right)$

For general metric spaces: if $f:M\to N$, (M,d_M), (N,d_N) metric spaces,

f not uniformly continuous $\Leftrightarrow$ $\exists$ seq. $x_n,y_n\in M$ with $d_M(x_n,y_n)<1/n$ and $d_N(f(x_n),f(y_n))\geq\varepsilon>0$.

Pg 203 #5) Define $g:[3,5]\to\mathbb{R}$ by $g(x)=\frac{f(x)}{x}$. Now g is cont. on $[3,5]$, diff. on $(3,5)$, so we can use mean value thm:

$g(3)=2$, $g(5)=2$ $\Rightarrow$ $\exists\, t\in[3,5]$ with $g'(t)=\frac{2-2}{2}=0$.

But $g'(t)=\frac{tf'(t)-f(t)}{t^2}=0$ $\Rightarrow$ $f'(t)=\frac{f(t)}{t}$ $\Rightarrow$ tangent line goes through $(0,0)$.

$A \subseteq \mathbb{R}^n$

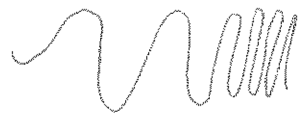
Pg 232 #12) $f: A \to \mathbb{R}^m$ is L-Lipschitz

a) If f is L-Lipschitz, then

$$\|x-y\| < \frac{\varepsilon}{L} \Rightarrow \|f(x)-f(y)\| < \varepsilon$$

$\therefore$ f unif. cont.

b) e.g. $f(x) = \sin(x^2)$



c) If f, g are L-Lipschitz, then $f+g$ is $2L$-Lipschitz:

$$\|(f(x)+g(x)) - (f(y)+g(y))\| \le \|f(x)-f(y)\| + \|g(x)-g(y)\| \le L\|x-y\| + L\|x-y\|$$
$$= 2L \cdot \|x-y\|.$$

d) If f, g are unif. cont, then $f+g$ is unif. cont:

let $\varepsilon > 0$; pick $\delta > 0$ small enough that if $\|x-y\| < \delta$ we have $\|f(x)-f(y)\| < \frac{\varepsilon}{2}$, $\|g(x)-g(y)\| < \frac{\varepsilon}{2}$. Then

$$\|(f(x)+g(x)) - (f(y)+g(y))\| \le \|f(x)-f(y)\| + \|g(x)-g(y)\| < \varepsilon.$$

For product, both claims are false: $x^2 = x \cdot x$ is a counterexample to both.

e) If f' is continuous, it is bounded on $[a,b]$: let $M > 0$ be a bound: $|f'(x)| \le M$ for all $x \in [a,b]$.

Now if $x, y \in [a,b]$, then $f(x) - f(y) = f'(t) \cdot (x-y)$ for some $t \in [a,b]$.

$$\Rightarrow |f(x) - f(y)| \le M|x-y| \Rightarrow f \text{ is } M\text{-Lipschitz}.$$

29) Suppose $f: \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le |x-y|^2$ for all $x, y \in \mathbb{R}$.

Let $x, y \in \mathbb{R}$, $x < y$. def: $x_0 = x$, $x_0 < x_1 < \dots < x_n = y$ such that $|x_{i+1} - x_i| = \frac{|x-y|}{n}$ (even steps from x to y)

Then $$|f(x) - f(y)| \le \sum_{i=1}^{n} |f(x_{i+1}) - f(x_i)| \le \sum_{i=1}^{n} |x_{i+1} - x_i|^2 \le n \cdot \left(\frac{|x-y|}{n}\right)^2 = \frac{|x-y|^2}{n}$$

as $n \to \infty$, we see that $|f(x) - f(y)| = 0$ so $f(x) = f(y)$.