Some solutions to HW12

Problem 1) $f: [0,1] \to \ell^\infty(\mathbb{N})$ defined by
$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \delta_{\pi(x)} & \text{if } x \in \mathbb{Q} \end{cases}$$

Here $\pi: \mathbb{Q} \cap [0,1] \to \mathbb{N}$ is any map that gives a different natural number to every rational number in $[0,1]$.

If $n \in \mathbb{N}$, the sequence $\delta_n \in \ell^\infty(\mathbb{N})$ is the sequence $(0, 0, \dots, \underset{(n)}{1}, 0, 0 \dots)$ that has 1 in n'th position and 0 elsewhere.

We claim that f is integrable and $\int_0^1 f(x)dx = 0 \in \ell^\infty(\mathbb{N})$.

For this, let $\underset{\shortparallel}{x_0}{}_{0} < x_1 < \dots < \underset{\shortparallel}{x_n}{}_{1}$ be a partition of $[0,1]$ and $c_i \in [x_{i-1}, x_i]$ for $i = 1, \dots, n$. Then the Riemann sum
$$S_f(\bar{x}, \bar{c}) = \sum_{i=1}^n (x_i - x_{i-1}) f(c_i)$$
is a linear combination of the values $f(c_i)$, which are either 0 or δ_{n_i}, where all the δ_{n_i} are different. Because we are using the sup-norm, we see
$$\| S_f(\bar{x}, \bar{c}) \|_\infty = \left\| \sum_{i=1}^n (x_i - x_{i-1}) f(c_i) \right\|_\infty \leq \max_{1 \leq i \leq n} (x_i - x_{i-1}) = \delta.$$
We can go to finer partitions such that $\delta \to 0$, and we see that $S_f(\bar{x}, \bar{c}) \to 0$ so $\int_0^1 f(x)dx = 0$.

If we look at the function $g: [0,1] \to \mathbb{R}$ def. by $g(x) = \|f(x)\|_\infty$, then g is the function
$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$
This is the same as example 4.8.2.

Pg 236 #44) Let $\underset{=0}{x_0} < x_1 < \cdots < \underset{=1}{x_n}$ be any partition of $[0,1]$.

By assumption, there are points $c_i \in [x_{i-1}, x_i]$ such that $f(c_i) = 0$. Then the Riemann sum $\sum_{i=1}^{n} (x_i - x_{i-1}) f(c_i) = 0$. But when we take finer partitions, the Riemann sum should converge to $\int_0^1 f(x)dx$ no matter what the c_i are, so the integral must be zero: $\int_0^1 f(x)dx = 0$.

If f is continuous, and $a \in [0,1]$ is any point, pick $c_n \in [a, a+\frac{1}{n}]$ such that $f(c_n) = 0$. Then $\lim_{n\to\infty} c_n = a$, so $f(a) = \lim_{n\to\infty} f(c_n) = \lim_{n\to\infty} 0 = 0$. Thus $f = 0$ on $[0,1]$.

Pg 244 #1) I claim that f_n converges uniformly on $[0,1]$ to $f(x) = x^2$: we have

$$|f_n(x) - f(x)| = \left|(x-\tfrac{1}{n})^2 - x^2\right| = \left|-\tfrac{2x}{n} + \tfrac{1}{n^2}\right| = \tfrac{1}{n}\left|-2x+\tfrac{1}{n}\right| \le \tfrac{2}{n}$$

for all $x \in [0,1]$, and as $n \to \infty$, the difference goes to 0 uniformly.

#2) The function $f_n(x) = x - x^n$ converges pointwise to

$$f(x) = \begin{cases} x & \text{if } x < 1 \\ 0 & \text{if } x = 1. \end{cases}$$

As the limit is not continuous, the convergence cannot be uniform. (It would contradict Prop 5.1.4)

Pg 244 #3) I think the answer is yes: let $\varepsilon > 0$.

First, pick n large enough such that $|f_n(x) - f(x)| < \varepsilon/3 \quad \forall x \in \mathbb{R}$.

Then pick $\delta > 0$ such that $\forall x, y : |x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon/3$.

Now if $x, y \in \mathbb{R}$ and $|x-y| < \delta$, then

$$
\begin{aligned}
|f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\
&\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
&< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

So f is uniformly continuous.

Pg 316 #2

a) Conv. uniformly to 0.

b) Conv. pointwise to 0, not uniformly.

c) Conv. uniformly to 0: note $\left|\frac{x}{kx+1}\right| < \frac{1}{k}$ for $x \in (0,1)$.

e) Conv. uniformly to $(1,0)$.

d) Conv. uniformly to 0: note $\left|\frac{x}{1+kx^2}\right| \leq \min\left\{ |x|, \frac{1}{k|x|} \right\}$.

Let $\varepsilon > 0$, choose $k > \frac{1}{\varepsilon^2}$.

Then if $|x| < \varepsilon$, then $\left|\frac{x}{1+kx^2}\right| < \varepsilon$, ✓

if $|x| \geq \varepsilon$, then $\left|\frac{x}{1+kx^2}\right| \leq \frac{1}{k|x|} < \frac{1}{\frac{1}{\varepsilon^2} \cdot \varepsilon} = \varepsilon$ ✓.