# Some solutions for HW13

**Problem 1)** Assume $\mathcal{F}$ is a set of functions $M \to N$ equicontinuous on $K \subseteq M$, which is compact.

Claim: $\mathcal{F}$ is uniformly equicontinuous, i.e.

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \ \forall f \in \mathcal{F} \quad \begin{pmatrix} d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon \\ (\forall x,y \in K) \end{pmatrix}$$

Proof: Let $\varepsilon > 0$. Since $\mathcal{F}$ equicont. on $K$, for each $x \in K$ we can pick $\delta_x > 0$ s.t.

$$\forall f \in \mathcal{F}, \forall y \in D(x, \delta_x) : \ d(f(x), f(y)) < \frac{\varepsilon}{2}.$$

Since the disks $D(x, \frac{\delta_x}{2})$ for $x \in K$ cover all of $K$, and $K$ is compact, we can find a finite subcover: $D(x_1, \frac{\delta_1}{2}) \cup \dots \cup D(x_n, \frac{\delta_n}{2}) \supseteq K$.

Denote $\delta = \min\{\frac{\delta_1}{2}, \dots, \frac{\delta_n}{2}\}$. Let's check that $\delta$ works:

If $x, y \in K$ satisfy $d(x,y) < \delta$, then $x \in D(x_i, \frac{\delta_i}{2})$ for some $i$.

Since $d(x,y) < \delta \le \frac{\delta_i}{2}$, we have $d(y, x_i) < \delta_i$ so $x, y \in D(x_i, \delta_i)$.

Then by the choice of $\delta_i$ we get

$$\forall f \in \mathcal{F} : \ \left. \begin{array}{l} d(f(x), f(x_i)) < \frac{\varepsilon}{2} \\ d(f(y), f(x_i)) < \varepsilon/2 \end{array} \right\} \Rightarrow \boxed{d(f(x), f(y)) < \varepsilon}$$

Thus $\mathcal{F}$ is uniformly equicontinuous.

Pg 272 #1) Let $B = \{ f \in C_b(\mathbb{R},\mathbb{R}) \mid f(x) > 0 \ \forall x \in \mathbb{R} \}$.

$B$ is not open: e.g. $f(x) = \frac{1}{1+x^2}$, $f \notin \mathrm{int}(B)$.

For any $\varepsilon > 0$, $f - \varepsilon \notin B$ but $f - \varepsilon$ is close to $f$.

In fact

$$\mathrm{int}(B) = \{ f \in C_b(\mathbb{R},\mathbb{R}) \mid \exists c > 0 : \forall x \in \mathbb{R} : f(x) \geq c \}.$$

#2) Let $D = \{ f \in C_b(\mathbb{R},\mathbb{R}) \mid f(x) \geq 0 \ \forall x \in \mathbb{R} \}$. $\supseteq B$

$D$ is closed since it is intersection

$$D = \bigcap_{x \in \mathbb{R}} \{ f \in C_b(\mathbb{R},\mathbb{R}) \mid f(x) \geq 0 \},$$ each of the "$f(x) \geq 0$" is a closed condition.

Thus $D \supseteq \mathrm{cl}(B)$. Also every $f \in D$ is a limit of functions in $B$, e.g. $f = \lim_{n \to \infty} (f + \frac{1}{n})$.

Thus $\underline{D = \mathrm{cl}(B)}$.

Pg 274 #3) Consider the function $I : C_b([0,1], \mathbb{R}) \to \mathbb{R}$ def. by

a) $I(f) = \int_0^1 f(x)dx$. This is a continuous linear map:

For $f, g \in C_b$ we have $|I(f-g)| = |I(f) - I(g)| \leq \|f-g\|_\infty$.

Thus the set $\{ f \in C_b \mid \int_0^1 f(x)dx \in (0,3) \} = I^{-1}((0,3))$ is open.

b) This is simply a rephrasing of the fact that if $f_n$ are continuous and $f_n \to f$ uniformly, then $f$ is continuous.

(Note that: $f_n \to f$ uniformly $\Leftrightarrow \|f_n - f\|_\infty \to 0$

$\Leftrightarrow f_n \to f$ in $C_b$.)

Pg 274 #4) As $[0,1]$ is compact, $\mathcal{B} \subseteq C([0,1], \mathbb{R})$ is closed, bounded and equicontinuous, we know that $\mathcal{B}$ is compact. We saw above in #3b) that the map $I: C([0,1], \mathbb{R}) \to \mathbb{R}$ is continuous, we know that in the compact set $\mathcal{B}$ it achieves its maximum value.

#5) Define a set $\mathcal{B} \subseteq C([a,b], \mathbb{R})$ by

$$\mathcal{B} = \left\{ F:[a,b] \to \mathbb{R} \;\middle|\; \begin{array}{l} F \text{ is } M\text{-Lipschitz}, \\ |F(x)| \le (b-a)M \;\; \forall x \in [a,b] \end{array} \right\}$$

Let $M > 0$ be an upper bound on all $f_n$: $|f_n(x)| \le M \;\; \forall n, \forall x \in [a,b]$

One can check that $\mathcal{B}$ is closed.
Clearly $\mathcal{B}$ is bounded, and since all $f \in \mathcal{B}$ are $M$-Lipschitz, we know that $\mathcal{B}$ is equicontinuous.
Thus $\mathcal{B}$ is compact.
We can also see that all $F_n \in \mathcal{B}$. Since $\mathcal{B}$ is compact, the sequence $(F_n)$ has a subsequence that converges in $\mathcal{B} \subseteq C([a,b], \mathbb{R})$. But convergence in $\|\cdot\|_\infty$ means uniform convergence so we are done.