1 Some problems from last semester

1. Suppose that \( f_k(x), k = 1, \ldots, \infty \) is a sequence of continuous functions on \([0,1]\) which converge pointwise to a continuous function \( f(x) \). Must they converge uniformly? (Proof or counterexample.)

2. Define \( f(x,y) \) for \((x,y) \in \mathbb{R}^2\) by \( f(0,0) = 0 \) and
\[
f(x,y) = \frac{x^2y}{\sqrt{|x|^3 + |y|^3}}
\]
for \((x,y) \neq (0,0)\). Are the partial derivatives \( \partial f / \partial x \) and \( \partial f / \partial y \) well defined at \((0,0)\)? Is \( f(x,y) \) differentiable at \((0,0)\)?

3. Give an example of a bounded function \( f : [0,1] \to \mathbb{R} \) which is Riemann integrable but which is discontinuous at an infinite number of points.

4. Let \( C[0,1] \) be the set of real-valued continuous functions on \([0,1]\) with the \( \sup \) norm. Give an example of a subset which is closed in the \( \sup \) norm and which is pointwise compact but which is not equicontinuous.

2 Some problems from the book

Do the following: (i) p. 334 #4; (ii) p. 344 #5; (iii) p. 355 # 5 (In the second part, justify your answer); (iv) p. 367 #6; (v) p. 388 #37.

3 Some problems on the delta function

1. Show that \( \delta(ax) = \delta(x)/|a| \). Hint: Consider \( \int \delta(ax) \, d(ax) \); remember that \( \delta(x) = \delta(-x) \).
2. Show that
\[ \delta(f(x)) = \sum_i \frac{\delta(x_i - x)}{|df/dx_i|} \]
where the \( x_i \) are the zeros of \( f(x) \) and \( df/dx_i \) is the derivative of \( f \) evaluated at \( x_i \). Hint: Expand \( f(x) \) near each zero in a Taylor series and use the preceding. You may assume that \( f \) is a polynomial but it is not necessary.

3. Recall that the Fourier transform \( \hat{f} \) of a function \( f \) of one variable is given by
\[ \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) \, dx, \]
and that its inverse is given by
\[ f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(xk) \, dk. \]
(Note. Conventions vary. Some authors place \( 1/2\pi \) in front of the first integral and omit the \( 1/\sqrt{2\pi} \) from before the second and some do the reverse; I prefer the present “balanced” form.) Show that
\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk = \delta(x-x'). \]
We saw that the expression of \( \delta(x) \) (extended periodically) as a Fourier series is
\[ \delta(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx}. \]
Show that the sum on the right is \( (C,1) \) summable to 0 for \( x \) not a multiple of \( 2\pi \). (p. 572 #2)

4. Show that in analogy with the formula for the delta function as a Fourier series, with the Fourier transform we have
\[ \delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk. \]

5. What is the analog of the previous summability assertion in the present case? Does it hold?

4 A problem on the web

Go to the site www.efunda.com/math/Laguerre/index.cfm (or get to it through Google by searching on “‘Laguerre polynomials’ +complete”), find an egregious error, and compose an appropriate short letter to the webmaster.