The digamma function and explicit permutations of the alternating harmonic series.

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Abstract

The main goal is to present a countable family of permutations of the natural numbers that provide explicit rearrangements of the alternating harmonic series and that we can easily define by some closed expression. The digamma function presents its ubiquity in mathematics once more by being the key tool in computing explicitly the simple rearrangements presented in this paper. The permutations are simple in the sense that composing one with itself will give the identity. We show that the countable set of rearrangements presented are dense in the reals. Then, slight generalizations are presented. Finally, we reprove a result given originally by J.H. Smith in 1975 that for any conditionally convergent real series guarantees permutations of infinite cycle type give all rearrangements of the series [4]. This result provides a refinement of the well known theorem by Riemann (see e.g. Rudin [3] Theorem 3.54).

1 Introduction

A permutation of order \( n \) of a conditionally convergent series is a bijection \( \phi \) of the positive integers \( \mathbb{N} \) with the property that \( \phi^n = \phi \circ \cdots \circ \phi \) is the identity on \( \mathbb{N} \) and \( n \) is the least such. Given a conditionally convergent series, a natural question to ask is whether for any real number \( L \) there is a permutation of order 2 (or \( n > 1 \)) such that the rearrangement induced by the permutation equals \( L \). This turns out to be an easy corollary of [4], and is reproved below with elementary methods. Other “simple” rearrangements have been considered elsewhere, such as in Stout [5] and the comprehensive references therein. On the road to answering the question above, before finding Smith’s paper, the alternating harmonic series was used as an archetypal easily manipulated conditionally convergent series. In particular, under conditions on positive integers \( a, b, c, \) and \( d \) that guarantee the below function is a bijection, we considered an explicit class of permutations of order 2 given by

\[
\phi(n) = \begin{cases} 
\frac{a n - b}{c} + d & \text{if } n \equiv b \mod a \\
\frac{a n - d}{c} + b & \text{if } n \equiv d \mod c \\
n & \text{otherwise.}
\end{cases}
\]
For example, if $a = 2, b = 1, c = 4, d = 2$, $\phi$ would swap odd integers with their double and leave all other naturals fixed. Such permutations of the alternating harmonic series gave new series that were analyzed by grouping $2ac$ consecutive terms together. Although there are many other permutations of the alternating harmonic series that one can find closed expressions for, these seem to be particularly simple to imagine. Moreover, they are interesting because when applied to the alternating series, the sums of these permutations are dense in $\mathbb{R}$, and none of these permutations of the alternating harmonic series diverges.

The digamma function, 

$$\psi : (0, \infty) \to \mathbb{R} \text{ defined by } \psi(x) = \frac{d}{dx} \log(\Gamma(x)),$$

where $\Gamma(x)$ is the gamma function defined on reals and $\log$ is the natural logarithm, proves to be a key tool in computing the explicit rearrangements of the alternating harmonic series given by the above permutations.

The main result will be explicitly computing rearrangements of the alternating harmonic series induced by the permutations above. Namely, we have the following result:

**Theorem 1.** Let $a, b, c, d$ be positive integers satisfying

- $b < a$,
- $d < c$, and
- $\gcd(a, c)$ does not divide $d - b$.

Let $\phi : \mathbb{N} \to \mathbb{N}$ be defined by

$$\phi(n) = \begin{cases} 
\frac{c^{n-b}}{a} + d & \text{if } n \equiv b \pmod{a} \\
\frac{a^{n-d}}{c} + b & \text{if } n \equiv d \pmod{c} \\
n & \text{otherwise.}
\end{cases}$$

Then $\phi$ is a bijection of the natural numbers and denoting $a_n = (-1)^{n+1}/n$,

$$\sum_{n=1}^{\infty} a_{\phi(n)} = \log(2) + \log(c/a)\left((-1)^b + (-1)^{a+b} \frac{2a}{2c}\right) + \log(a/c)\left((-1)^d + (-1)^{c+d} \frac{2c}{2a}\right).$$

An obvious generalization of the permutations above to permutations of order $n$ are considered below and are also be explicitly computable. We start by proving necessary lemmas, move on to the main results, and conclude with some digressions.

# Results

## 2.1 Important lemmas

We start with a result relating the convergence of the series $\sum_{i=0}^{\infty} d_i$ with the convergence of the series $\sum_{i=0}^{\infty} \sum_{j=0}^{n-1} d_{ni+j}$. It turns out that if $d_i \to 0$ as $i \to \infty$,
then the sums are equal. In particular, in some cases we can take a conditionally convergent series and turn it into an absolutely convergent series by summing over what will be referred to as blocks.

**Lemma 1.** Let \( \{d_i\}_{i=0}^{\infty} \) be a sequence such that \( d_i \to 0 \) as \( i \to \infty \). Fix \( n \in \mathbb{N} \).

\(
\sum_{i=0}^{\infty} d_i \) converges if and only if \( \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} d_{ni+j} \) converges and in this case they converge to the same limit.

**Proof.** Assume \( \sum_{i=0}^{\infty} d_i = D \). By definition of convergence of a series, we see that by grouping \( n \) many \( d_i \)'s at a time, the grouped series

\[
\sum_{i=0}^{\infty} (d_{ni} + d_{ni+1} + \ldots + d_{ni+n-1}) = \sum_{i=0}^{n-1} d_{ni+j}
\]

converges to the same limit.

On the other hand, assume \( \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} d_{ni+j} = D \). Let \( \varepsilon > 0 \).

Let \( M_1 \) be such that

\[
m \geq M_1 \implies |d_i| < \frac{\varepsilon}{2n}.
\]

Let \( M_2 \) be such that

\[
m \geq M_2 \implies \left| \left( \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} d_{ni+j} \right) - D \right| < \frac{\varepsilon}{2}
\]

Then, for \( M > n + \max\{nM_2, M_1\} \) we have

\[
\left| \sum_{i=0}^{M} d_i - D \right| \leq \left| \sum_{i=0}^{M-(M \mod n)} d_i - D \right| + \left| \sum_{i=(M-(M \mod n)) + 1}^{M} d_i \right|
\]

\[
= \left| \sum_{i=0}^{(M-(M \mod n))/n} \left( \sum_{j=0}^{n-1} d_{ni+j} \right) - D \right| + \left| \sum_{i=(M-(M \mod n)) + 1}^{M} d_i \right|
\]

\[
\leq \left| \sum_{i=0}^{(M-(M \mod n))/n} \left( \sum_{j=0}^{n-1} d_{ni+j} \right) - D \right| + \sum_{i=(M-(M \mod n)) + 1}^{M} |d_i|.
\]

Now, in the second sum there are

\[
M - ((M - (M \mod n)) + 1) = (M \mod n) - 1 < n
\]

many terms. Moreover, for \( i > (M - (M \mod n)) \) we have

\[
|d_i| < \frac{\varepsilon}{2n}
\]

since
\[ (M - (M \mod n)) + 1 > M_1 + n - (M \mod n) + 1 \geq M_1. \]

Furthermore, \[ (M - (M \mod n))/n \geq nM_2/n = M_2 \]
implies
\[ \left| \left( \sum_{i=0}^{n-1} d_{n+i} \right) - D \right| < \frac{\varepsilon}{2}. \]

Hence, the last sum in the chain of inequalities above is less than \( \varepsilon \). Therefore \( \sum_{i=0}^{\infty} d_i \) converges to \( D \). \( \square \)

We provide three key properties of the digamma function that will be needed later on.

**Lemma 2** (Some properties of digamma function). Let \( \psi \) be the digamma function for positive real argument as defined above. Let \( a, b, N, \) and \( q \) be natural numbers.

(i) \[ \sum_{k=0}^{N-1} \frac{1}{ak + b} = \frac{\psi(b/a + N) - \psi(b/a)}{a}. \]

This could be easily deduced from Gronwall and Jensen[2], p.138 equation (19).

(ii) As \( x \to \infty \),
\[ \psi(x) = \log(x) + O(1/x). \]

This can also be found in [2], implied by equations (16) on p.135 and (28) p.142.

(iii) \[ \sum_{p=0}^{q-1} \psi(a + p/q) = q(\psi(qa) - \log(q)). \]

This is proven in [2], p.140 equation (23).

**Lemma 3.** Let \( a_i, b_i \) be naturals and let \( c_i \) be integers for \( 1 \leq i \leq n \). Assume \[ \frac{c_1}{a_1} + \cdots + \frac{c_n}{a_n} = 0. \]

Then
\[ \sum_{k=0}^{\infty} \frac{c_1}{a_1k + b_1} + \cdots + \frac{c_n}{a_nk + b_n} \]

converges absolutely to
\[-\sum_{i=1}^{n} c_i \frac{\psi(b_i/a_i)}{a_i}.\]

Proof. We first show that the sum converges absolutely. For fixed $k$ we have

\[
\frac{c_1}{a_1 k + b_1} + \cdots + \frac{c_n}{a_n k + b_n} = \frac{\sum_{i=1}^{n} c_i (a_1 k + b_1) \cdots (a_i k + b_i) \cdots (a_n k + b_n)}{(a_1 k + b_1) \cdots (a_n k + b_n)}.
\]

Considering the numerator and denominator as polynomials in $k$, the coefficient of $k^n$ in the denominator is \(\prod_{i=1}^{n} a_i\), the coefficient of $k^n$ in the numerator is 0, and the coefficient of $k^{n-1}$ in the numerator is also 0 by the assumptions placed on the $a_i$ and $c_i$:

\[
\sum_{i=1}^{n} c_i a_1 \cdots \hat{a}_i \cdots a_n = \sum_{i=1}^{n} c_i \frac{a_1 \cdots a_n}{a_i} = (a_1 \cdots a_n) \sum_{i=1}^{n} \frac{c_i}{a_i} = 0.
\]

Applying the ratio test to the series above with the convergent series \(\sum_{k=1}^{\infty} 1/k^2\), this part of the proof is complete.

Next, we use property (i) of $\psi$ in Lemma 2 to see that

\[
\sum_{k=0}^{N-1} \sum_{i=1}^{n} \frac{c_i}{a_i k + b_i} = -\sum_{i=1}^{n} \frac{c_i}{a_i} \psi\left(b_i/a_i\right) + \sum_{i=1}^{n} \frac{c_i}{a_i} \psi\left(b_i/a_i + N\right).
\]

Therefore all we need to show is that

\[
\sum_{i=1}^{n} \frac{c_i}{a_i} \psi\left(b_i/a_i + N\right) \to 0 \text{ as } N \to \infty.
\]

By property (ii) in Lemma 2,

\[
\sum_{i=1}^{n} \frac{c_i}{a_i} \psi\left(b_i/a_i + N\right) = \sum_{i=1}^{n} \frac{c_i}{a_i} \log\left(b_i/a_i + N\right) + O(1/N).
\]

However, since \(\sum_{i=1}^{n} c_i/a_i = 0\), the latter sum is equal to

\[
\sum_{i=1}^{n} \left(\frac{c_i}{a_i} \log\left(b_i/a_i + N\right) - \frac{c_i}{a_i} \log(N)\right) + O(1/N)
\]

\[= \sum_{i=1}^{n} \frac{c_i}{a_i} \log\left(b_i/a_i + N\right) - \log(N) + O(1/N).
\]

Letting $N \to \infty$, we see that the numerators in the sum above vanish, as does the $O(1/N)$ term. This completes the proof. \(\square\)
2.2 The digamma function and rearrangements of the alternating harmonic series

Now we move on to the proof of Theorem 1, restated below for convenience.

**Theorem 1.** Let \( a, b, c, d \) be positive integers satisfying

- \( b < a, \)
- \( d < c, \) and
- \( \gcd(a, c) \) does not divide \( d - b. \)

Let \( \phi : \mathbb{N} \to \mathbb{N} \) be defined by

\[
\phi(n) = \begin{cases} 
\frac{c^{n-b}}{a} + d & \text{if } n \equiv b \mod a \\
\frac{a^{n-d}}{c} + b & \text{if } n \equiv d \mod c \\
n & \text{otherwise.}
\end{cases}
\]

Then \( \phi \) is a bijection of the natural numbers and

\[
\sum_{n=1}^{\infty} \frac{(-1)^{\phi(n)+1}}{\phi(n)} = \log(2) + \log(c/a) \frac{(-1)^b + (-1)^{a+b}}{2a} + \log(a) \frac{(-1)^d + (-1)^{c+d}}{2c}.
\]

**Proof.** We first check that \( \phi \) is a well-defined bijection. Let \( g = \gcd(a, c) \). Then

\[
d - b = ak - ck' = g(ak/g - ck'/g).
\]

Hence, the only way \( ak + b = ck' + d \) for some \( k, k' \in \mathbb{N} \) is if \( g|d - b. \) Therefore \( \phi \) is well-defined. Therefore, since \( ak + b \) never equals to \( ck' + d, \) we can split \( \mathbb{N} \) into a disjoint union: those elements that are \( b \mod a, \) those that are \( d \mod c \) and those that are neither. Using this decomposition it is easy to see that \( \phi \) is a bijection since \( ak + b \) came from (and went to) \( ck + d \) for all \( k \geq 0. \)

We now split \( \mathbb{N} \) into conjugacy classes modulo \( 2ac: 2ack + 1, ..., 2ack + 2ac. \) We choose \( 2ac \) because it is even and is divisible by \( g. \) The latter condition is necessary to capture everything that is going on in the partial sums, but the former is for convenience. For each \( k \) there are \( 2c \) terms in the set \( \{2ack + n + 1 : 0 \leq n \leq 2ac - 1\} \) that are \( b \mod a: \)

\[
2ack + b, 2ack + a + b, 2ack + 2a + b, ..., 2ack + (2c - 1)a + b,
\]

and there are \( 2a \) terms that are \( d \mod b: \)

\[
2ack + d, 2ack + c + d, 2ack + 2c + d, ..., 2ack + (2a - 1)c + d.
\]

The philosophy behind computing rearrangements is to consider partial sums of the original series of height \( N, \) see which terms are brought into the partial sum by the permutation, which are taken out, and which ones are not moved.
Let’s consider the partial sums of the alternating harmonic series and split the partial sums into blocks:

\[
\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} = \sum_{k=0}^{\infty} \sum_{n=0}^{2ac-1} \frac{(-1)^{2ack+n}}{2ack + n + 1} = \sum_{k=0}^{\infty} \sum_{n=0}^{2ac-1} \frac{(-1)^n}{2ack + n + 1}.
\]

This notation will only be for convenience because we are not actually considering an infinite series different from the alternating harmonic series. Now consider the blocks inside the series. We reorganize them into an even more convenient form for the purpose of explicitly seeing how our permutation \( \phi \) acts on our series, and note that we can consider each block separately in order to see what is going on with the series as a whole:

\[
\sum_{n=0}^{2ac-1} \frac{(-1)^n}{2ack + n + 1} = \sum_{n=0}^{2c-1} \frac{(-1)^{na+b+1}}{2ack + na + b} + \sum_{n=0}^{2a-1} \frac{(-1)^{nc+d+1}}{2ack + nc + d} + \sum_{n=0}^{2ac-1} \frac{(-1)^n}{2ack + n + 1}.
\]

Out of the five sums, the first two sums are those that are moved away when we apply the permutation \( \phi \), and the remaining terms are those that do not change. Note that

\[
\phi(2ack + na + b) = 2a^2k + nc + d,
\]

\[
\phi(2ack + nc + d) = 2a^2k + na + b.
\]

Hence, the partial sum of the rearranged series will contain the terms

\[
\sum_{n=0}^{2c-1} \frac{(-1)^{nc+d+1}}{2c^2k + nc + d} + \sum_{n=0}^{2a-1} \frac{(-1)^{na+b+1}}{2a^2k + na + b} + \sum_{n=0}^{2ac-1} \frac{(-1)^n}{2ack + n + 1}.
\]

First, let us make sure that we have indeed permuted all the terms of our series, as it might not be clear since we grouped the terms in blocks of \( 2ac \). For every \( k' \) there are unique \( n \) and \( k \) such that \( 2ack + n + 1 = ak' + b \), and unique \( n \) and \( k \) such that \( 2ack + n + 1 = ck' + d \). This is because we can find a unique \( k \)
with the property that \( k' \in [2ack + 1, 2ac(k + 1)] \). The rest follows since \( \phi \) is a bijection.

Next, we claim that the sum of the coefficients of all terms in our new blocks is 0; if this is the case, we will be able to consider a new infinite series that is absolutely convergent by Lemma 3, and then we will be able to compute the rearranged series by applying Lemma 2 identity (iii). We will show that the sum we obtain is the same as the original rearranged sum with Lemma 1 and the proof will be complete.

The claims are as follows:

\[
(I) \sum_{n=0}^{2a-1} \frac{(-1)^{na+b+1}}{2a^2} + \sum_{n=0}^{2c-1} \frac{(-1)^{na+b}}{2ac} = 0,
\]

\[
(II) \sum_{n=0}^{2c-1} \frac{(-1)^{nc+d+1}}{2c^2} + \sum_{n=0}^{2a-1} \frac{(-1)^{nc+d}}{2ac} = 0,
\]

and

\[
(III) \sum_{n=0}^{2ac-1} \frac{(-1)^n}{2ac} = 0.
\]

Indeed, we see:

if \( a \) is odd, \((I) = 0 \) since both sums are identically 0;

if \( a \) is even, \( (I) = \frac{2a(-1)^{b+1}}{2a^2} + 2c(-1)^b = \frac{(-1)^{b+1}}{a} - \frac{(-1)^{b+1}}{a} = 0. \)

With an identical computation, \((II) = 0 \) as well. Finally,

\[
(III) = \frac{1}{2ac} \cdot \frac{1 - (-1)^{2ac}}{1 - (-1)} = 0
\]

since

\[
\sum_{n=0}^{N-1} x^n = \frac{1 - x^N}{1 - x}.
\]

We may now apply Lemma 3 to compute our rearranged series in block form. We forget about the sum of \( \sum_{n=0}^{2ac-1} \frac{(-1)^n}{2ac+1} \) blocks, as we know summing these blocks over \( k \) gives \( \log(2) \) by Lemma 1. Lemma 3 tells us the rearranged series, excluding the term we are ignoring, gives us:

\[
\sum_{n=0}^{N-1} x^n = \frac{1 - x^N}{1 - x}.
\]
\[
\sum_{n=0}^{2c-1} \frac{(-1)^{nc+d}}{2c^2} \psi\left(\frac{(nc + d)/2c^2}{2}\right)
\]

(1)

\[
+ \sum_{n=0}^{2a-1} \frac{(-1)^{na+b}}{2a^2} \psi\left(\frac{(na + b)/2a^2}{2}\right)
\]

(2)

\[
+ \sum_{n=0}^{2c-1} \frac{(-1)^{na+b+1}}{2ac} \psi\left(\frac{(na + b)/2ac}{2}\right)
\]

(3)

\[
+ \sum_{n=0}^{2a-1} \frac{(-1)^{nc+d+1}}{2ac} \psi\left(\frac{(nc + d)/2ac}{2}\right).
\]

(4)

Hence,

\[
(1) = \frac{(-1)^d}{2c^2} \sum_{n=0}^{2c-1} (-1)^{nc} \psi\left(\frac{n}{2c} + \frac{d}{2c^2}\right)
\]

\[
= \frac{(-1)^d}{2c^2} \left( \sum_{n=0}^{c-1} \psi\left(\frac{n}{c} + \frac{d}{2c^2}\right) + \sum_{n=0}^{c-1} (-1)^c \psi\left(\frac{2n + 1 + d}{2c^2}\right) \right),
\]

where we just split the sum into \(n\) even and \(n\) odd, respectively. Then, the above equals

\[
\frac{(-1)^d}{2c^2} \left( \sum_{n=0}^{c-1} \psi\left(\frac{n}{c} + \frac{d}{2c^2}\right) + \sum_{n=0}^{c-1} (-1)^c \psi\left(\frac{2n + 1 + d}{2c^2}\right) \right)
\]

\[
= \frac{(-1)^d}{2c^2} \left( \sum_{n=0}^{c-1} \psi\left(\frac{n}{c} + \frac{d}{2c^2}\right) + (-1)^c \sum_{n=0}^{c-1} \psi\left(\frac{n}{c} + \frac{d}{2c^2} + \frac{1}{2}\right) \right).
\]

We now apply identity (iii) of the digamma function, namely

\[
\sum_{p=0}^{q-1} \psi(a + p/q) = q(\psi(qa) - \log(q)),
\]

to see that the sum (1) is actually equal to

\[
\frac{(-1)^d}{2c^2} \left( c\left( \psi\left(\frac{d}{2c}\right) - \log(c) \right) + c(-1)^c \left( \psi\left(\frac{d}{2c} + \frac{1}{2}\right) - \log(c) \right) \right)
\]

\[
= \frac{(-1)^d}{2c} \left( \psi\left(\frac{d}{2c}\right) + (-1)^c \psi\left(\frac{d}{2c} + \frac{1}{2}\right) + \log(c)((-1)^{c+1} - 1) \right).
\]
With identical computations we can show

\[
(2) = \frac{(-1)^b}{2a} \psi\left(\frac{b}{2a}\right) + (-1)^a \psi\left(\frac{b}{2a} + \frac{1}{2}\right) + \log(a)((-1)^{a+1} - 1),
\]

\[
(3) = \frac{(-1)^{b+1}}{2a} \psi\left(\frac{b}{2a}\right) + (-1)^a \psi\left(\frac{b}{2a} + \frac{1}{2}\right) + \log(c)((-1)^{a+1} - 1),
\]

\[
(4) = \frac{(-1)^{d+1}}{2c} \psi\left(\frac{d}{2c}\right) + (-1)^c \psi\left(\frac{d}{2c} + \frac{1}{2}\right) + \log(a)((-1)^{c+1} - 1).
\]

Summing each of these, we get the miraculous reduction

\[
(1) + (2) + (3) + (4) = \log(c/a)\left(\frac{(-1)^b}{2a} + \frac{(-1)^{a+b}}{a} + \log(a/c)\frac{(-1)^d + (-1)^{c+d}}{2c}\right).
\]

Hence, by Lemma 1, rearranged alternating harmonic series is also equal to

\[
\log(2) + \frac{1}{2}\left(\log(c/a)\frac{(-1)^b}{a} + \log(a/c)(-1)^d + (-1)^{c+d}\right)
\]

since

\[
\sum_{k=1}^{\infty} a_{\phi(k)} = \sum_{k=0}^{\infty} \sum_{n=0}^{2ac-1} a_{\phi(2ack+n+1)} = \sum_{k=0}^{2c-1} \sum_{n=0}^{2a-1} \frac{(-1)^{n+1} + (-1)^{n+1}}{2a^2k + nc + d} + \sum_{n=0}^{2a-1} \frac{(-1)^{na+b+1}}{2a^2k + nc + b} + \sum_{n=0}^{2c-1} \frac{(-1)^{na+b}}{2ack + na + b} + \sum_{n=0}^{2a-1} \frac{(-1)^{nc+d}}{2ack + nc + d} + \sum_{n=0}^{2a-1} \frac{(-1)^{n+1}}{2ack + n + 1}.
\]

2.3 Examples

Example (i) We know that if \( a = c \), the sum should not change because we are not moving the terms far enough away no matter the choice of \( b \) or \( d \).

Indeed letting \( a = c \) in the formula above gives \( \log(2) \) since \( \log(a/c) = 0 = \log(c/a) \) in this case.

Example (ii) Although each \( \phi \) takes the alternating harmonic series to another convergent series, the new sums are unbounded in the reals.

Letting \( a = 2, b = 1, c = 2^m, d = 2 \), we see these integers satisfy the
conditions of Theorem 1. The new sum is
\[
\log(2) + \frac{1}{2} \left( \log(2^{m-1}) - \frac{2}{2} + \log(2^{1-m}) \frac{2}{2^m} \right)
\]
\[
= \log(2) + \frac{1}{2} \left( (1-m) \log(2) + (1-m) \log(2) \frac{1}{2^{n-1}} \right)
\]
\[
= \log(2) + \frac{(1-m) \log(2)}{2} (1 + 2^{1-m}) \to -\infty
\]
as \(m \to \infty\).

In particular, these types of rearrangements are unbounded from below. The intuition was that we sent negative terms very far away, and switched them with the small positive terms. Similarly, letting \(a = 2, b = 2, c = 2^m, d = 1\), the rearranged series sums to
\[
\log(2) + \frac{(m-1) \log(2)}{2} \left( 1 + 2^{1-m} \right) \to \infty
\]
as \(m \to \infty\).

Hence, these types of rearrangements are also unbounded above. Again, the intuition here is that we sent positive terms very far away, and brought very small negative terms to the front.

Example (iii) If \(a\) and \(c\) are both odd, then
\[
\log(2) + \frac{1}{2} \left( \frac{\log(c/a)}{a} (-1)^b + (-1)^{a+b} c + \log(a/c) (-1)^d + (-1)^{c+d} \right)
\]
\[
= \log(2) + \frac{1}{2} \left( \frac{\log(c/a)}{a} \cdot 0 + \log(a/c) \cdot 0 \right) = \log(2).
\]

Although the rearrangement involving the digamma function may seem mysterious, there is a nice way to see the fact above by simply subtracting the partial sums of the rearranged series and the original series and showing that this difference tends to 0 in the limit.

Since we always had a \(\log(2)\) term in the sum of the rearranged series, i.e., the old sum was always a summand of the closed expression for the new sum, an important trick to understand is that we need to consider not just the partial sums of the rearranged series, but the difference of partial sums of the original and the rearranged series. Let \(\phi : \mathbb{N} \to \mathbb{N}\) be a bijection. We split the difference of the sums into convenient pieces:
\[
\sum_{n=1}^{N} a_{n} - \sum_{n=1}^{N} a_{\phi(n)} \\
= \sum_{\phi(n) > N}^{\phi(n) > N} a_{n} + \sum_{\phi(n) \leq N}^{\phi(n) \leq N} a_{n} - \sum_{\phi(n) > N}^{\phi(n) > N} a_{\phi(n)} - \sum_{\phi(n) \leq N}^{\phi(n) \leq N} a_{\phi(n)} \\
= \sum_{\phi(n) > N}^{\phi(n) > N} a_{n} - a_{\phi(n)}.
\]

This was the important tool used to prove Theorem 1. We use this to show that in the case where \( \phi \) swaps \( ak + b \) and \( ck + d \), with \( a, c \) both odd, the rearranged series has the same sum. WLOG assume \( a < c \). We compute:

\[
\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{N} \frac{(-1)^{\phi(n)+1}}{\phi(n)} \\
= \sum_{\phi(n) > N}^{\phi(n) > N} \frac{(-1)^{n+1}}{n} - \sum_{\phi(n) > N}^{\phi(n) > N} \frac{(-1)^{\phi(n)+1}}{\phi(n)} \\
= \sum_{\left\lfloor \frac{N}{a} \right\rfloor}^{\left\lfloor \frac{N}{a} \right\rfloor} \frac{(-1)^{an+b+1}}{an+b} - \sum_{\left\lfloor \frac{N}{c} \right\rfloor}^{\left\lfloor \frac{N}{c} \right\rfloor} \frac{(-1)^{cn+d+1}}{cn+d},
\]

where all the terms that were not congruent to \( b \mod a \) nor to \( d \mod c \) got subtracted off. Next, since

\[
(-1)^{an+b+1} = (-1)^{n+b+1} \quad \text{and} \quad (-1)^{nc+d+1} = (-1)^{n+d+1},
\]
as both \( a \) and \( c \) are odd, the above is equal to

\[
\sum_{\left\lfloor \frac{N}{a} \right\rfloor}^{\left\lfloor \frac{N}{a} \right\rfloor} \frac{(-1)^{n+b+1}}{an+b} - \sum_{\left\lfloor \frac{N}{c} \right\rfloor}^{\left\lfloor \frac{N}{c} \right\rfloor} \frac{(-1)^{n+d+1}}{cn+d} \\
= \sum_{\left\lfloor \frac{N}{a} \right\rfloor}^{\left\lfloor \frac{N}{a} \right\rfloor} \frac{(-1)^{n+b+1}}{an+b} - \sum_{\left\lfloor \frac{N}{a} \right\rfloor}^{\left\lfloor \frac{N}{a} \right\rfloor} \frac{(-1)^{n+d+1}}{cn+d} \\
= \sum_{\left\lfloor \frac{N}{a} \right\rfloor}^{N} \frac{((-1)^{b+1}c - (-1)^{d+1}a)n + ((-1)^{b+1}d - (-1)^{d+1}b)}{(an+b)(cn+d)} \\
+ \sum_{\left\lfloor \frac{N}{a} \right\rfloor}^{N} \frac{((-1)^{b+1}d - (-1)^{d+1}b)}{(an+b)(cn+d)}
\]

In the last step, we broke up the partial sum into the terms with an \( n \) in the numerator and those without. This helps us see that the second
series of partial sums converges to 0 as $N \to \infty$ by applying the squeeze theorem with $\frac{d+b}{(an+b)(cn+d)}$:

$$\lim_{N \to \infty} \left| \sum_{\lceil N/a \rceil}^{N} (-1)^b \left( \frac{(-1)^{b+1}d - (-1)^{d+1}b}{(an+b)(cn+d)} \right) \right| \leq \lim_{N \to \infty} \sum_{\lceil N/a \rceil}^{N} \frac{d+b}{(an+b)(cn+d)} = 0.$$  

Moreover, $\sum_{n=1}^{\infty} (-1)^{n} \frac{n((-1)^{b+1}c - (-1)^{d+1}a)}{(an+b)(cn+d)}$ converges by the alternating series test, as for $k > 0$,

$$\frac{d}{dx} \left( \frac{xk}{(ax+b)(cx+d)} \right) = \frac{k(acx^2 + x(ad + bc) + bd) - kx(2acx + ad + bc)}{(acx^2 + x(ad + bc) + bd)^2} = \frac{-acx^2 + bd}{\text{something positive}}.$$  

But $bd - acx^2 < 0$ for $x > \sqrt{bd/4ac}$. But this square root is always less than 1 since $b < a, d < c$. So actually $\left\{ \frac{kn}{(an+b)(cn+d)} \right\}_{n=1}^{\infty}$ is a decreasing sequence for any $k \in \mathbb{N}$. Therefore $\sum_{n=1}^{\infty} (-1)^{n} \frac{kn}{(an+b)(cn+d)}$ converges as claimed, so by the alternating series test,

$$\lim_{N \to \infty} \sum_{\lceil N/a \rceil}^{N} (-1)^{n} \frac{kn}{(an+b)(cn+d)} = 0.$$  

For $k$ negative, we see that

$$\lim_{N \to \infty} \sum_{\lceil N/a \rceil}^{N} (-1)^{n+1} \frac{-kn}{(an+b)(cn+d)} = 0$$

again by the alternating series test. Since $(-1)^{b+1}c - (-1)^{d+1}a$ is either positive or negative, we see that the two cases above imply

$$\left| \sum_{\lceil N/a \rceil}^{N} (-1)^{n} \frac{n((-1)^{b+1}c - (-1)^{d+1}a)}{(an+b)(cn+d)} \right| \to 0.$$  

This gives an interesting class of permutations that one might not expect to force the rearranged alternating harmonic series to keep the same sum as the original series.

Example (iv) Now we give an example of a sum that is not given by a rearrangement of this sort, namely 0. To get 0 as the sum of such a rearrangement, we require

$$\log(c/a) \left( \frac{(-1)^b}{a} + \frac{(-1)^{a+b}}{c} \right) + \log(a/c) \left( \frac{(-1)^d}{c} + \frac{(-1)^{c+d}}{a} \right) = -2 \log(2).$$
We split this possibility into cases:

(1) Assume that \(a\) is even and \(c\) is odd. Then the expression on the left-hand side above simplifies to

\[
\log(c/a) \frac{2(-1)^b}{a}.
\]

This equals to \(-2\log(2)\) iff

\[
|\log(c/a)| = a \log(2).
\]

Since we assumed \(a\) is even, we must have \(a < 2^a c\) for all \(a, c \in \mathbb{N}\).

(2) Simply switch \(a\) and \(c\) above to get the result for \(a\) odd and \(c\) even.

(3) Assume that \(a\) and \(c\) are both even and without loss of generality \(a < c\). Then the above expression simplifies to

\[
\log(c/a) \left( \frac{(-1)^b}{a} - \frac{(-1)^d}{c} \right) = -\log(2)
\]

\[
\implies \log(c/a)(-1)^b c - (-1)^d a = -ac \log(2).
\]

Since \(c > a\), we see that \(b\) cannot be even as the left-hand side will be positive. So we must actually have

\[
\log(c/a)(c + (-1)^d a) = ac \log(2).
\]

I.e.,

\[
2^{ac} = \left( \frac{c}{a} \right)^{c+a}.
\]

In particular, \(c = ma\) for some \(m \in \mathbb{N}\). So \(2^{2m} = m^{a(m+1)}\). Taking the \(amth\) root, \(2^a = m^{1+1/m}\). But \(m^{1/m}\) is not rational for \(m > 1\). Contradiction.

(4) If \(a\) and \(c\) are both odd, Example (iii) showed that equality cannot hold since the left-hand side is identically 0.

Example (v) By the computation in the proof of Theorem 1, the following identity is true:

\[
\log(c/a) \frac{(-1)^b + (-1)^{a+b} + \log(a/c)(-1)^d + (-1)^{c+d}}{2a} = \frac{1}{2c} \sum_{n=0}^{2c-1} \frac{(-1)^{nc+d}}{c} \psi\left( \frac{nc + d}{2c^2} \right) - \frac{(-1)^{na+b}}{a} \psi\left( \frac{na + b}{2ac} \right)
\]

\[
+ \frac{1}{2a} \sum_{n=0}^{2a-1} \frac{(-1)^{na+b}}{a} \psi\left( \frac{na + b}{2a^2} \right) - \frac{(-1)^{nc+d}}{c} \psi\left( \frac{nc + d}{2ac} \right).
\]
Example (vi) We also have the identity
\[
\log(2) = \sum_{k=0}^{2m-1} (-1)^k \psi(k/2m)
\]
for \(m \in \mathbb{N}\), as we may choose a nice splitting of the alternating harmonic series into blocks and applying Lemma 3 along with Lemma 1.

Example (vii) Although we only have a countable set of outputs, we can show that the range of the partial function \(f : \mathbb{N}^4 \rightarrow \mathbb{R}\), where \(f(a, b, c, d)\) is defined by the sum of the rearrangement \(ak + b \iff ck + d\), as long as \(a, b, c, d\) meet the conditions of Theorem 2, is dense around \(\log(2)\).

Let \(p\) be an odd prime. Let \(a = p, b = 1\) and \(c = 2p\). Then
\[
f(a, b, c, d) = \log(2) + \frac{(-1)^d + 1}{2p} \log(2).
\]

We can make \(p\) as large as we want. Even better, we have the following corollary to Theorem 1:

**Corollary 1.** Rearranged sums of the alternating harmonic series given in Theorem 1 are dense in \(\mathbb{R}\). Namely, given \(a, b, c, d\) positive integers with \(b < a, d < c\) and \(d - b\) not divisible by \(\text{gcd}(a, c)\), the image of the function \(f(a, b, c, d)\) defined by
\[
f(a, b, c, d) = \log(2) + \frac{(-1)^b + (-1)^a + b}{2a} \log(c/a) \frac{(-1)^d + (-1)^c + d}{2d},
\]
where \(a, b, c, d\) satisfy the conditions of Theorem 1, is dense in \(\mathbb{R}\).

**Proof.** First, we prove that
\[
\frac{\log(6m + 3)}{6n} - \frac{\log(6n)}{6n}, m, n \in \mathbb{N}
\]
is dense in \((0, \infty)\).

To show this, fix a positive \(L \in \mathbb{R}\) and let \(\varepsilon > 0\).
There is \(N_1 \in \mathbb{N}\) such that \(n > N_1 \implies |\log(6n)/6n| < \frac{\varepsilon}{2}\).
There is \(\delta > 0\) such that \(|\log(1 + x)| < \varepsilon\) whenever \(|x| < \delta\).
There is \(N_2 \in \mathbb{N}\) such that \(n > N_2 \implies 4e^{-6nL} < \delta\).
Let \(n > \max\{N_1, N_2\}\).

By the pigeonhole principle, there is \(m \in \mathbb{N}\) such that
\[
e^{6nL} - 4 \leq 6m + 3 \leq e^{6nL} + 4.
\]

Hence,
\[
\log(e^{6nL} - 4) \leq \log(6m + 3) \leq \log(e^{6nL} + 4).
\]
This implies
\[ 6nL + \log(1 - 4e^{-6nL}) \leq \log(6m + 3) \leq 6nL + \log(1 + 4e^{-6nL}). \]
Combining this with the inequalities
\[ \log(1 - 4e^{-6nL}) > -\varepsilon \quad \text{and} \quad \log(1 + 4e^{-6nL}) < \varepsilon, \]
we conclude
\[ 6nL - \varepsilon < \log(6m + 3) < 6nL + \varepsilon \implies \left| \frac{\log(6m + 3)}{6n} - L \right| < \frac{\varepsilon}{6}. \]
Hence,
\[
\left| \frac{\log(6m + 3)}{6n} - \frac{\log(6n)}{6n} - L \right| \\
\leq \left| \frac{\log(6m)}{6n} \right| + \left| \frac{\log(6m + 3)}{6n} - L \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{6} < \varepsilon.
\]
Next, given positive \( L \) and \( \varepsilon \) as above, let \( a = 6m + 3 \) and \( c = 6n \) as above. Then we see that \( \gcd(a, c) = 3 \). So let \( b = 1 \) and \( d = 2 \), so that \( a, b, c, d \) match all the conditions necessary in the hypotheses of Theorem 1. We have
\[
\frac{\log(c/a)}{2a} \cdot (-1)^b + (-1)^{a+b} + \log(a/c)\frac{(-1)^d + (-1)^{c+d}}{2c} \\
= \log(a/c)\frac{(-1)^d}{c},
\]
since \( a \) odd and \( c \) is even. Now use the fact that \( d = 2 \) and split up \( \log(a/c) = \log(a) - \log(c) \) to make the above quantity equal to
\[
\frac{\log(a)}{c} - \frac{\log(c)}{c} = \frac{\log(6m + 3)}{6n} - \frac{\log(6n)}{6n}.
\]
Hence,
\[
\left| f(a, b, c, d) - (L + \log(2)) \right| \\
= \left| \log(2) + \left( \frac{\log(c/a)}{2a} \cdot (-1)^b + (-1)^{a+b} + \log(a/c)\frac{(-1)^d + (-1)^{c+d}}{2c} \right) - (L + \log(2)) \right| \\
= \left| \log(2) + \frac{\log(6m + 3)}{6n} - \frac{\log(6n)}{6n} - (L + \log(2)) \right| \\
= \left| \frac{\log(6m + 3)}{6n} - \frac{\log(6n)}{6n} - L \right| < \varepsilon.
\]
Since $L > 0$ was arbitrary, and we see that $f(a, b, c, d)$ can be arbitrarily close to any $L + \log(2)$, we conclude that $f(a, b, c, d)$ is dense in $(\log(2), \infty)$. To show density in $(-\infty, \log(2))$, use the simple identity
\[ f(a, 1, c, 2) = -f(a, 2, c, 1). \]

\[ \square \]

### 2.4 Permutations of order $n$ of the alternating harmonic series

A slight generalization of Theorem 1, considering instead nontrivial permutations of order $n$, is the following:

**Theorem 2.** Let $a_i, b_i$ be positive integers satisfying

- $b_i < a_i$ for $1 \leq i \leq m$,
- $\gcd(a_i, a_j)$ does not divide $b_i - b_j$ for distinct $1 \leq i, j \leq m$.

If we permute the elements of the alternating harmonic series by
\[ a_1k + b_1 \implies a_2k + b_2 \implies \cdots \implies a_{m-1}k + b_{m-1} \implies a_mk + b_m \implies a_1k + b_1, \]
i.e., if we define $\phi : \mathbb{N} \rightarrow \mathbb{N}$ by
\[ \phi(n) = \begin{cases} a_2 \frac{n-b_1}{a_1} + b_2 & \text{if } m \equiv b_1 \mod a_1 \\
 \cdots & \\
 a_n \frac{n-b_{n-1}}{a_{n-1}} + b_n & \text{if } m \equiv b_{m-1} \mod a_{m-1} \\
 a_1 \frac{n-b_{m}}{a_{m}} + b_1 & \text{if } m \equiv b_{m} \mod a_{m} \\
 n & \text{otherwise}, \end{cases} \]
then this permutation is well-defined, is a bijection of the natural numbers, and
\[ \sum_{k=1}^{\infty} \frac{(-1)^{\phi(k)+1}}{\phi(k)} = \log(2) + \sum_{i=1}^{m} \log \left( a_i \frac{a_{i+1}}{a_{i+1} + b_{i+1}} \right) \left[ (-1)^{b_{i+1}} + (-1)^{a_{i+1}+b_{i+1}} \right], \]
where $a_{m+1}$ is defined to be $a_1$ and similarly $b_{m+1} = b_1$.

The proof is very similar to the proof of Theorem 1. We can also generalize the series a little bit by considering arbitrary arithmetic progressions in the denominator:
2.5 Permutations of order 2 of conditionally convergent real series

We move on to the main theorems.

**Theorem 2.** Given a conditionally convergent series of reals $\sum_{n=1}^{\infty} a_n$ and $L \in \mathbb{R}$, there is a permutation $\phi$ of order 2 such that $\sum_{n=1}^{\infty} a_{\phi(n)} = L$.

**Proof.** Let $A$ be such that $A = \sum_{n=0}^{\infty} a_n$. If $L = A$ the proof is trivial. First we show the result for $L < A$. Fix $B > 0$. Find $\{a_n\}_{j=0}^{\infty}$, a subsequence of $\{a_n\}_{n=0}^{\infty}$, consisting only of positive reals, satisfying

1. $\sum_{j=0}^{\infty} a_{n_j} = \infty$.
2. Given $m \in \mathbb{N}$, if $k_m$ is the least such that $\sum_{j=m}^{k_m} a_{n_j} > B$, there is some $l_m$ such that $n_{k_m} < l_m < n_{k_m+1}$ and $|a_{n_j}| \leq 2^{-m}$ for all $n \geq l_m$.

Such a subsequence exists because $\sum_{n=0}^{\infty} a_n$ converges conditionally; in particular, the sum of the positive terms diverges to $\infty$ and the sum of the negative terms diverges to $-\infty$. However, since the original series converges, the terms must converge to 0. Denote $\{a_{n_j}\}$ by $\{b_j\}$ and $\{a_{l_j}\}$ by $\{c_j\}$.

Define a bijection $\phi : \mathbb{N} \to \mathbb{N}$ by swapping $l_j$ and $n_j$, and leaving all other $n$ fixed. This is a permutation of order 2 since there is no $l_m$ that equals some $n_j$.

Let $\varepsilon > 0$. Let $N$ be such that $m > N$ implies

1. $k > m > N \Rightarrow \sum_{n=m+1}^{k} a_n < \varepsilon/4$.
2. $|b_m| < \varepsilon/8$.
3. $\sum_{n=m}^{\infty} |c_n| < \varepsilon/8$.

Fix $\ell = n_{k_{N+1}}$. Then for any $m \geq \ell$, we have $n_{k_j} \leq m < n_{k_{j+1}}$ for a unique $j > N$. More precisely, we have $n_{k_{j+d}} \leq m < n_{k_{j+d+1}}$ for some $0 \leq d < k_{j+1} - k_j$. We have the equality

$$\left| \sum_{n=0}^{m} a_{\phi(n)} - A + B \right| = \left| \sum_{n=0}^{m} a_n - \sum_{n=j}^{k_{j+d}} b_n + \sum_{n=j}^{k_{j+d}} c_n - A + B + \delta_{0d}(b_j - c_j) \right|,$$

as the rearranged series equals the original series minus the terms moved far away, plus the terms moved closer. Indeed either $b_j, ..., b_{k_{j+d}}$ or $b_{j+1}, ..., b_{k_{j+d}}$ are the only terms moved away since $n_{k_j} < l_j < n_{k_{j+1}}$. That is, if we consider the partial sum up to $m$ such that $n_{k_j} \leq m < n_{k_{j+1}}$, then we move away at most $b_1, ..., b_{k_{j+d}}$. However, $b_1, ..., b_{j-1}$ get sent no further than to the $n_{k_{j-1}}$st place and $b_j$ is sent somewhere between $n_{k_j}$ and $n_{k_{j+1}}$. Hence, if $d = 0$ then...
b_j is moved further than the range we are summing over, but if d > 0 then it
is not. The same argument is why we have the sum over c_j's moved within the
region j ≤ n ≤ k_j + d. Using the triangle inequality and (1) − (3), we have

\[ \left| \sum_{n=n_{k_j}+1}^{m} a_n \right| + \sum_{n=0}^{n_{k_j}} a_n - \sum_{n=j}^{k_j+d} b_n + \sum_{n=0}^{k_j+d} c_n - A + B + \delta_{bd}(|b_j| + |c_j|) \]

\[ < \varepsilon/4 + \sum_{n=0}^{n_{k_j}} a_n - A + |B - \sum_{n=j}^{k_j} b_n| + \sum_{n=k_j+1}^{k_j+d} b_n + \sum_{n=0}^{k_j+d} c_n + \varepsilon/4. \]

To see that the above is less than

\[ \varepsilon/4 + \varepsilon/8 + |b_{k_j}| + |b_j| + \varepsilon/8 + \varepsilon/4 = \varepsilon, \]

notice that b_j + ... + b_{k_j} > B where k_j was the least such, hence

\[ B + b_{k_j} \geq b_j + \ldots + b_{k_j} > B. \]

I.e. the difference \( |B - \sum_{n=j}^{k_j} b_n| \) is bounded by \( |b_{k_j}| \). Again, since \( b_j + \ldots + b_{k_j} > B \) is the least such, \( b_n \)'s are all positive, and \( d < k_{j+1} - k_j \), the term \( |\sum_{n=k_j+1}^{k_j+d} b_n| \) is bounded above by \( |b_j| \).

So the rearrangement converged to \( L = A - B \). \( B > 0 \) was arbitrary, so we can rearrange the series with a permutation of order 2 to converge to any \( L < A \). To show that we can rearrange the series to converge to any \( L > A \), just take \( b_n \) to be negative with their sum diverging to infinity, and the similar adjustments for \( c_n \).

An important observation from the proof of Theorem 1 is the following:

**Corollary 2.** Given a conditionally convergent series of reals \( \sum_{n=0}^{\infty} a_n \), an infinite cycle type \( s = (s_1, \ldots, s_k, \ldots) \), and \( L \in \mathbb{R} \), there is a permutation \( \phi : \mathbb{N} \to \mathbb{N} \) with cycle type \( s \) so that

\[ \sum_{n=0}^{\infty} a_{\phi(n)} = L. \]

**Proof.** In the construction above, choose \( n_j \)'s and \( l_m \)'s so that

\[ n_{k_m} < l_m < l_m + s_m < n_{k_m+1} \]

and

\[ n > l_m \implies |a_n| < 2^{-(s_1 + \ldots + s_m)}. \]

Modify the construction of the sequences and the permutation in the following way: instead of simply permuting \( a_{n_j} \) and \( a_{l_j} \), permute

\[ l_j \implies n_j \implies l_j + s_j - 1 \implies l_j + s_j - 2 \implies \cdots \implies l_j + 1 \implies l_j. \]
Hence, we brought $a_{t_j}$, something small, closer; moved $a_{n_j}$, something relatively big, further; and permuted $s_j - 1$ many consecutive terms that won’t affect anything. An almost identical elementary estimate shows that the rearranged series will converge to $L = A - B$ as it did above. With the exception of the two key summands already singled out in Theorem 1, each $s_j$-cycle permutes our series trivially with respect to the structure of our summation. This was almost like cheating, but was sufficient to prove the corollary.

Another slight modification will show that given any infinite cycle type, any conditionally convergent series, and any $-\infty \leq \alpha \leq \beta \leq \infty$, we can permute our series so that the lim inf of partial sums equal $\alpha$, and the lim sup of the partial sums will equal $\beta$. This will provide the desired generalization of Riemann’s summation theorem from [3].

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