On the formality of derived intersections

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Abstract

In this short note we study the derived intersection of two smooth subvarieties of a smooth variety and we give a necessary and sufficient criterion for the intersection to be formal. As a consequence we obtain a derived base change theorem for non-transversal intersections. We also sketch an application to the study of the derived fixed locus of a finite group action.

1. Introduction

1.1. Let S be a smooth variety S over a field of characteristic zero and let X and Y be smooth subvarieties of S. We shall assume that X and Y intersect cleanly (meaning that their scheme theoretic intersection $W = \mathsf{X} \times_{\mathsf{S}} \mathsf{Y}$ is smooth) but not necessarily transversely. Derived algebraic geometry associates to this data a geometric object, the *derived intersection* of X and Y,

$$W' = \mathsf{X} \times^{\mathsf{R}}_{\mathsf{S}} \mathsf{Y}.$$

It is a differential graded (dg) scheme whose structure complex is constructed by taking the derived tensor product of the structure sheaves of X and Y. (The reader unfamiliar with the subject of dg schemes is referred to Section **??**.) The underived intersection W naturally sits inside W' as a closed subscheme.

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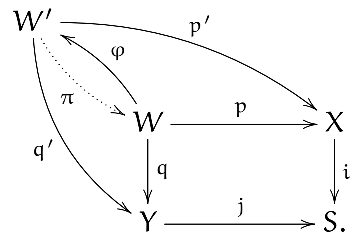
We organize these spaces and the maps between them in the diagram



The purpose of this note is to understand when W' is as simple as possible. Our main result (Theorem 1.8) makes this precise in two ways. In algebraic terms it gives a necessary and sufficient criterion for W' to be *formal* in the sense of [**?**]. In geometric terms it gives a necessary and sufficient condition for the existence of a map $\pi : W' \to W$ exhibiting W' as the total space of a shift $E[-1]$ of a vector bundle E over W,

$$W' = \mathrm{Tot}(E[-1]).$$

When this holds we gain a geometric understanding of the structure of the maps φ, p' and q': φ is the inclusion of the zero section of the bundle $\mathrm{Tot}\, E[-1]$, and p' and q' factor through the bundle map π.

1.2. The problem we study originates in classical intersection theory. While the scheme-theoretic intersection W is determined algebraically by the *underived* tensor product

$$\mathscr{O}_W = \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{O}_Y,$$

Serre [**?**] argued that in order to obtain a theory with good formal properties we need to use instead the *derived* tensor product

$$\mathscr{O}_{W'} = \mathscr{O}_X \otimes^{L}_{\mathscr{O}_S} \mathscr{O}_Y.$$

Since $\mathscr{O}_{W'}$ is naturally a commutative dg algebra we can regard it as the structure complex of a dg scheme W'.

1.3. For classical applications it suffices to work with the class of $\mathscr{O}_{W'}$ in K-theory. Put differently, we only need to know the sheaves.

$$\mathscr{H}^{-*}(\mathscr{O}_{W'}) = \mathrm{Tor}^S_*(\mathscr{O}_X, \mathscr{O}_Y).$$

A local calculation as in [**?**, Proposition A.3] shows that these sheaves are the exterior powers $\wedge^* E^\vee$ of the *excess bundle* E, the vector bundle on W defined by

$$E = \frac{T_S}{T_X + T_Y}.$$

(We omit writing the restrictions from X, Y, S to W in formulas like the one above. So when we write T_X we mean $T_X|_W$, the restriction of the tangent bundle of X to W.) The excess bundle E is a vector bundle on W of rank $\dim S + \dim W - \dim X - \dim Y$ which measures the failure of the intersection to be transversal.

1.4. For certain problems, however, it is not enough to know just the Tor sheaves $\mathscr{H}^k(\mathscr{O}_{W'})$; we need to understand the full dg algebra $\mathscr{O}_{W'}$. For example there is considerable interest in computing $\mathrm{Ext}^*_S(i_*F, j_*G)$ for vector bundles F and G on X and on Y. These groups can be computed using the spectral sequence

$$^2E^{pq} = H^p(W, F^\vee \otimes G \otimes \wedge^q E) \Rightarrow \mathrm{Ext}^{p+q}_S(i_*F, j_*G),$$

from [?, Theorem A.1]. (Again, we omit the restrictions of F and G to W.) The differentials in this spectral sequence arise as obstructions to splitting the canonical filtration on $\mathscr{O}_{W'}$, that is, they vanish if there is a isomorphism

$$\mathscr{O}_{W'} \cong \bigoplus_k \mathscr{H}^k(\mathscr{O}_{W'})[-k].$$

1.5. In the above discussion we have skated over an important detail. The splitting of the structure complex of a dg scheme is not an intrinsic property of the dg scheme; rather, the concept only makes sense for a morphism from a dg scheme to a base scheme.

Consider a dg scheme Z' over an ordinary scheme Z, i.e., a dg scheme endowed with a structure morphism $s : Z' \to Z$. We shall say that Z' is *formal* over Z if $s_*\mathscr{O}_{Z'}$ is split as an object of $\mathbf{D}(Z)$, in other words if there exists a formality isomorphism

$$s_*\mathscr{O}_{Z'} \cong \bigoplus_k \mathscr{H}^k(s_*\mathscr{O}_{Z'})[-k].$$

The terminology is inspired by [?], where it is proved that the de Rham complex of a compact Kähler manifold is formal (over the manifold itself).

1.6. The derived intersection $W' = X \times^R_S Y$ can be viewed as a dg scheme over several base schemes: either one of X, Y, S, or $X \times Y$ can naturally serve as an underlying scheme for W'. (However, note that in general we can not present W' as a dg scheme over W.) Our primary motivation for studying derived intersections comes from our desire to understand the degeneration of the spectral sequence in (1.4). For this purpose it is most useful to regard W' as a dg scheme over $X \times Y$. Indeed, in this approach the structure sheaf

$\mathscr{O}_{W'}$ of W' is the kernel of the functor $j^* i_* : \mathbf{D}(X) \to \mathbf{D}(Y)$ between dg enhancements $\mathbf{D}(X)$, $\mathbf{D}(Y)$ of the derived categories of X and Y. (We omit the $\mathbf{R}$'s and $\mathbf{L}$'s in front of derived functors for simplicity.)

1.7. Formality of $W'/X \times Y$ turns out to be closely related to properties of the inclusion $W \hookrightarrow W'$. We shall say that a map of spaces $W \to W'$ *splits* if it admits a left inverse. If $W \to W'$ is a closed embedding we shall say that it *splits to first order* if the induced map $W \to W'^{(1)}$ splits, where $W'^{(1)}$ is the first infinitesimal neighborhood of W inside W'.

The above concepts also make perfect sense for spaces (schemes, dg schemes) over a fixed base scheme, in which case we require the inverse map to be a map over the base scheme.

We are now ready to state the main theorem of the paper [1].

1.8. Theorem. *The following statements are equivalent.*

(1) *There exists an isomorphism of dg functors* $\mathbf{D}(X) \to \mathbf{D}(Y)$

$$j^* i_*(-) \cong q_*(p^*(-) \otimes \mathbb{S}(E^\vee[1])).$$

(2) W' *is isomorphic to* $\mathsf{Tot}_W E[-1]$ *as dg schemes over* $X \times Y$.

(3) W' *is formal as a dg scheme over* $X \times Y$.

(4) *The inclusion* $W \to W'$ *splits over* $X \times Y$.

(5) *The inclusion* $W \to W'$ *splits to first order over* $X \times Y$.

(6) *The short exact sequence*

$$0 \to T_X + T_Y \to T_S \to E \to 0$$

of vector bundles on W *splits.*

1.9. The above theorem can be seen as a generalization of several classical results: base change for flat morphisms or, more generally, for Tor-independent morphisms; the Hochschild-Kostant-Rosenberg isomorphism for schemes [**?**]; and the formality theorem for derived self-intersections of the first two authors [**?**].

[1] While working on the final draft of this paper the authors became aware that a closely related result was obtained independently and at about the same time by Grivaux [**?**]

1.10. However, the main application we have in mind for Theorem 1.8 is in the study of derived fixed loci. Let φ be a finite-order automorphism of a smooth variety Z. We are interested in the fixed locus W of φ,

$$W = Z^{\varphi} = \{z \in Z \mid \varphi(z) = z\}.$$

This fixed locus can be studied using intersection theory, as we can view W as the intersection (inside $Z \times Z$) of the diagonal Δ and the graph Δ^{φ} of φ,

$$W = \Delta \times_{Z \times Z} \Delta^{\varphi}.$$

1.11. This description makes it clear that the expected dimension of the fixed locus is zero. Whenever W is positive dimensional the cause is a failure of transversality of Δ and Δ^{φ}. It then makes sense to study the *derived fixed locus* of φ, W', which we define as the derived intersection

$$W' = \Delta \times^{R}_{Z \times Z} \Delta^{\varphi}.$$

The excess intersection bundle E for this problem is

$$E = (T_Z)_{\varphi} = \frac{T_Z}{\langle v - \varphi(v) \rangle},$$

the vector bundle on W obtained by taking coinvariants of $T_Z|_W$ with respect to the action of φ.

In this setup Theorem 1.8 allows us to get the following geometric characterization of the derived fixed locus W'.

1.12. Corollary. *The derived fixed locus W' is isomorphic, as a dg scheme over $Z \times Z$, to the total space over W of the dg vector bundle $(T_Z)_{\varphi}[-1]$,*

$$W' \cong \mathrm{Tot}_W\left((T_Z)_{\varphi}[-1]\right).$$

1.13. We apply the above result to the study of orbifolds. Let G be a finite group acting on a smooth variety Z, and let $\mathscr{X}$ be the quotient stack $[Z/G]$. We are interested in the relationship between the inertia stack of $\mathscr{X}$

$$I\mathscr{X} = \mathscr{X} \times_{\mathscr{X} \times \mathscr{X}} \mathscr{X}$$

and the free loop space of $\mathscr{X}$, which is the corresponding derived intersection

$$L\mathscr{X} = \mathscr{X} \times^{R}_{\mathscr{X} \times \mathscr{X}} \mathscr{X}.$$

As usual we have the maps $i, j, p, q, p', q', \varphi$ as in (1.1). Note that $i = j$ and $p = q$.

The formality of the derived fixed loci implies the following result.

1.14. Corollary. *There exists a map* $\pi : \mathsf{L}\mathscr{X} \to \mathrm{I}\mathscr{X}$ *presenting the free loop space* $\mathsf{L}\mathscr{X}$ *as the total space of a vector bundle over the inertia stack* $\mathrm{I}\mathscr{X}$*,*

$$\mathsf{L}\mathscr{X} = \mathsf{Tot}_{\mathrm{I}\mathscr{X}}(\mathsf{E}[-1]).$$

In particular, the two projections p' *and* q' *are equal and there is a natural isomorphism of dg functors*

$$q'_* p'^*(-) \cong - \otimes q'_* \mathscr{O}_{\mathsf{L}\mathscr{X}}.$$

1.15. In order to state the above result in more concrete terms (in particular, in order to describe the vector bundle E) we need the following notations for $g \in G$:

- Z^g is the (underived) fixed locus of g in Z;
- i_g is the closed embedding of Z^g in Z;
- c_g is the codimension of Z^g in Z;
- ω_g is the relative dualizing bundle of the embedding i_g, that is, the top exterior power of the normal bundle $N_{Z^g/Z}$ of Z^g in Z,

  $$\omega_g = \wedge^{c_g} N_{Z^g/Z};$$

- T_g is the vector bundle on Z^g obtained by taking coinvariants of $T_Z|_{Z^g}$ with respect to the action of g;
- Ω^j_g is the dual, along Z^g, of $\wedge^j T_g$;
- $\mathbb{S}(\Omega^1_g[1])$ is the symmetric algebra of $\Omega^1_g[1])$, i.e., the object of $\mathbf{D}(Z^g)$

  $$\mathbb{S}(\Omega^1_g[1]) = \oplus \Omega^j_g[j].$$

1.16. The inertia stack $\mathrm{I}\mathscr{X}$ is a global quotient stack, realized as the quotient

$$\mathrm{I}\mathscr{X} = [\mathrm{I}Z/G],$$

where $\mathrm{I}Z$ is the smooth, disconnected scheme

$$\mathrm{I}Z = \coprod_{g \in G} Z^g.$$

The action of $h \in G$ on Z^g maps it to $Z^{hgh^{-1}}$, and hence $\mathrm{I}\mathscr{X}$ is a smooth Deligne-Mumford stack whose connected components are in one-to-one correspondence with the conjugacy classes in G. The component of $\mathrm{I}\mathscr{X}$ corresponding to the conjugacy class $[g]$ is isomorphic to $[Z^g/C(g)]$ where $C(g)$ is the centralizer of g in G.

1.17. Corollary. *On the component* $[Z^g/C(g)]$ *of* $I\mathscr{X}$ *corresponding to the conjugacy class* $[g]$ *the bundle* E *is given by the* $C(g)$*-equivariant bundle* T_g*. Therefore the object* $q'_*\mathscr{O}_{L\mathscr{X}} \in \mathbf{D}(\mathscr{X})$ *is represented by the* G*-equivariant object of* $\mathbf{D}(Z)$

$$\bigoplus_{g\in G} i_{g,*}\,\mathbb{S}(\Omega_Z^g[1]).$$

The above result yields immediately a Hochschild-Kostant-Rosenberg isomorphism for orbifolds, generalizing results of of Baranovsky [**?**] and Ganter [**?**].

1.18. Corollary. *We have*

(1) $\Delta^*\Delta_*\mathscr{O}_{\mathscr{X}} = \bigoplus_{g\in G} i_{g,*}\mathbb{S}(\Omega_Z^g[1])$.

(2) $\mathsf{HH}_*(\mathscr{X}) = \left(\bigoplus_{g\in G}\bigoplus_{q-p=*} H^p(Z^g,\Omega_g^q)\right)_G$.

(3) $\mathsf{HH}^*(\mathscr{X}) = \left(\bigoplus_{g\in G}\bigoplus_{p+q=*} H^{p-c_g}(Z^g,\wedge^q T_g\otimes\omega_g)\right)^G$.

1.19. The paper is organized as follows. In Section **??** we collect some general results about dg schemes in the sense of Kapranov. In particular we discuss how a dg scheme W' presented over a base scheme S can be regarded as a dg scheme *over* S and we construct presentations of the derived intersection W' over X, Y, $X\times Y$, and S. In Section **??** we present the proof of Theorem 1.8. In the final section of the paper we discuss applications to orbifolds, and present proofs of Corollaries 1.12, 1.14, 1.17, and 1.18.

1.20. Conventions. We work over a field of characteristic zero. The same results also hold when the characteristic of the ground field is sufficiently large; we shall make it explicit in the statement of each theorem how large the characteristic needs to be for the results to hold. All schemes are assumed to be smooth, quasi-projective over this field.

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2. Background on dg schemes

In this section we review some facts from the basic theory of differential graded schemes, following the work of Ciocan-Fontanine and Kapranov [?]. We emphasize the point of view that a dg scheme $Z' = (Z, \mathscr{O}_{Z'})$ should be thought of as a dg scheme *over* Z, and explain how the derived intersection $W' = X \times^R_S Y$ can be viewed in a natural way as a dg scheme over X, Y, $X \times Y$, or S.

2.1. Following Ciocan-Fontanine and Kapranov [?], a *differential graded scheme* Z' is a pair $(Z, \mathscr{O}_{Z'})$ consisting of an ordinary scheme Z, the *base scheme* of Z', and a complex of quasi-coherent sheaves $\mathscr{O}_{Z'}$ on Z, the *structure complex* of Z'. The complex $\mathscr{O}_{Z'}$ is assumed to be endowed with the structure of a commutative dg algebra over $\mathscr{O}_Z$, and must satisfy

1. $\mathscr{O}^i_{Z'} = 0$ for $i > 0$;
2. $\mathscr{O}^0_{Z'} = \mathscr{O}_Z$.

Maps between dg schemes are obtained by a localization procedure similar to the one that leads to the construction of derived categories. In a first stage morphisms of dg schemes are considered as maps of ringed spaces. For dg schemes $Z' = (Z, \mathscr{O}_{Z'})$ and $W' = (W, \mathscr{O}_{W'})$ a morphism $Z' \to W'$ consists of a map of schemes $f : Z \to W$ along with a map of dg algebras $f^\# : f^* \mathscr{O}_{W'} \to \mathscr{O}_{Z'}$. In the resulting category we have a natural notion of quasi-isomorphisms of dg schemes – those morphisms $(f, f^\#)$ for which $f^\#$ is a quasi-isomorphism of complexes of sheaves. Formally inverting those quasi-isomorphisms produces a category $\mathfrak{DSch}$, the right derived category of schemes.

2.2. Because quasi-isomorphisms become isomorphisms in $\mathfrak{DSch}$, isomorphic dg schemes can be presented over different base schemes. Thus the base scheme is not an intrinsic part of a dg scheme in $\mathfrak{DSch}$. For certain purposes, however, it is useful to be able to refer to the base scheme of a dg scheme. Instead of carrying over this additional data, we give an alternative way of looking at the relationship between a dg scheme Z' and its supporting scheme Z.

The definition of dg schemes implies that the structure complex $\mathscr{O}_{Z'}$ of a dg scheme $Z' = (Z, \mathscr{O}_{Z'})$ admits a natural morphism of dg algebras $\mathscr{O}_Z \to \mathscr{O}_{Z'}$ (where $\mathscr{O}_Z$ is regarded as a complex concentrated in degree zero). This shows that a dg scheme Z' presented over a base scheme Z comes with a canonical morphism $Z' \to Z$.

2.3. This observation motivates us to study dg schemes *over a fixed scheme* Z instead of arbitrary dg schemes. These are dg schemes Z' endowed with a morphism $Z' \to Z$. (We shall mostly be concerned with the situation when this morphism is *affine* – this is the case when the dg scheme Z' is presented over Z. But the concept makes sense in general.) As in the theory of schemes, morphisms of dg schemes over Z are morphisms between dg schemes which commute with the structure morphisms.

2.4. We now turn to discussing the construction of derived intersections over various bases. We place ourselves in the context described in the introduction, with X and Y subschemes of S. The structure complex of the derived intersection $W' = X \times^{R}_{S} Y$ is obtained by taking the derived tensor product $\mathscr{O}_{W'} = \mathscr{O}_X \otimes^{\mathbf{L}}_{\mathscr{O}_S} \mathscr{O}_Y$.

The main question we want to address is over what base scheme should the complex $\mathscr{O}_{W'}$ be considered. If the schemes were affine, this would be equivalent to deciding whether to consider this tensor product as an algebra over $\mathscr{O}_X$, $\mathscr{O}_Y$, $\mathscr{O}_S$, etc. Likewise, in the general case there is no canonical choice of base scheme for the dg scheme W', and either one of X, Y, S, or $X \times Y$ can serve for this purpose. For example, it is easy to see W' as a dg scheme over X by resolving $\mathscr{O}_Y$ by a flat commutative dg algebra over S and pulling back the resolution to X. Similarly, in order to obtain a model over S resolve both $\mathscr{O}_X$ and $\mathscr{O}_Y$ over S and tensor them over $\mathscr{O}_S$.

It is essential to emphasize that in general it is not possible to present W' as a dg scheme over W, the underived intersection.

2.5. For the purpose of this article we are most interested in a model of W' whose base scheme is $X \times Y$. To obtain such a presentation define

$$\mathscr{O}_{W'} = \mathscr{O}_{\Gamma_i} \circ \mathscr{O}_{\Gamma_j} = \pi_{XY,*}(\pi^*_{XS}\mathscr{O}_{\Gamma_i} \otimes_{X\times SY\times Y} \pi^*_{SY}\mathscr{O}_{\Gamma_j}),$$

the convolution of the kernels $\mathscr{O}_{\Gamma_i} \in \mathbf{D}(X \times S)$ and $\mathscr{O}_{\Gamma_j} \in \mathbf{D}(S \times Y)$. Here $\Gamma_i \subset X \times S$, $\Gamma_j \subset S \times Y$ are the graphs of the inclusions $i : X \hookrightarrow S$, $j : Y \hookrightarrow S$, and π_{XS}, π_{SY} and π_{XY} are the projections from $X \times S \times Y$ to $X \times S$, $S \times Y$, and $X \times Y$, respectively. (We omit the $\mathbf{R}$'s and $\mathbf{L}$'s in front of derived functors for simplicity.) The reader can easily supply the required equality of tensor products of rings which shows that this definition of W' is quasi-isomorphic to the previous ones.

Note that the kernels $\mathscr{O}_{\Gamma_i}$ and $\mathscr{O}_{\Gamma_j}$ induce the functors $i_* : \mathbf{D}(X) \to \mathbf{D}(S)$ and $j^* : \mathbf{D}(S) \to \mathbf{D}(Y)$. Since $\mathscr{O}_{W'}$ is the convolution of these kernels, we conclude that $\mathscr{O}_{W'}$ is the kernel of the dg functor $j^*i_* : \mathbf{D}(X) \to \mathbf{D}(Y)$.

This fact allows us to connect with our earlier discussion in (1.4). Indeed, in order to guarantee the degeneration of the spectral sequence computing

$\mathsf{Ext}^*_S(i_*F, j_*G)$ we need to understand the functor j^*i_*. Since this functor is controlled by W' as presented over $X \times Y$, this explains why we want to understand formality properties of $W'/X \times Y$ and not over other bases.

2.6. There is another description of $\mathscr{O}_{W'}$ as an object in $\mathbf{D}(X \times Y)$ which is useful in the proof of Theorem 1.8. The original problem of studying the intersection of X and Y into S can be reformulated to study the intersection of $X \times Y$ with the diagonal in $S \times S$. Let $\bar{\imath}$ and $\bar{\jmath}$ be the embeddings of S and $X \times Y$ into $S \times S$.

The derived and underived intersections in the new problem are the same as in the old one. The excess bundle is also the same. However, by replacing the original problem with the new one we have simplified the initial situation in two ways. First, the embedding $\bar{\imath} : S \hookrightarrow S \times S$ is now split. Second, since the object $\bar{\jmath}^*\bar{\imath}_*\mathscr{O}_S$ realizes $\mathscr{O}_{W'}$ as an object of $\mathbf{D}(X \times Y)$, the problem of understanding the functor j^*i_* is replaced by the problem of understanding the single object $\bar{\jmath}^*\bar{\imath}_*\mathscr{O}_S$. We have replaced the functor j^*i_* by the more complicated functor $\bar{\jmath}^*\bar{\imath}_*$, but we only apply it to a single object $\mathscr{O}_S$ which is well behaved.

2.7. We now turn to questions of formality. Given a dg scheme Z' over a scheme Z, with structure morphism $f : Z' \to Z$, we shall say that Z' is *formal* over Z if $f_*\mathscr{O}_{Z'}$ is formal as an object in $\mathbf{D}(Z)$, i.e., if there exists an isomorphism

$$f_*\mathscr{O}_{Z'} \cong \bigoplus_j \mathscr{H}^j(f_*\mathscr{O}_{Z'})[-j]$$

of objects in $\mathbf{D}(Z)$.

2.8. The notion of formality of a dg scheme depends on the scheme over which we are working. Indeed, consider a smooth subvariety X of a smooth space S, and let $X' = X \times^R_S X$ be the derived self-intersection of X inside S. Then X' is a dg scheme over X in two distinct ways (using the two projections), and hence it is also a dg scheme over $X \times X$. In [**?**] the first two authors introduced two classes,

$$\alpha_{HKR} \in H^2(X, N \otimes N^\vee \otimes N^\vee)$$

and

$$\eta \in H^1(X, T_X \otimes N^\vee).$$

The results of [loc. cit.] and the present paper show that

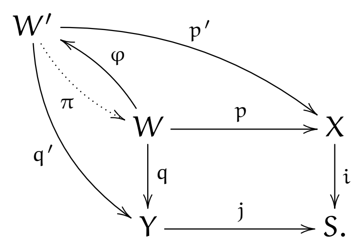
– X' is formal over X if and only if the HKR class α_{HKR} vanishes;

– X' is formal over $X \times X$ if and only if the class η, vanishes.

It is known ([?]) that $\eta = 0$ implies $\alpha_{HKR} = 0$, but not vice-versa. Thus X' being formal over $X \times X$ implies it is formal over X, but the converse can fail.

3. The proof of the main theorem

In this section we shall prove the main theorem 1.8, which we restate below. We place ourselves in the context of (1.1), with X and Y smooth subschemes of S, and with W' and W being their derived and underived intersections, respectively. The maps between these spaces are listed in the diagram below



The excess intersection bundle E on W is defined as

$$E = \frac{T_S}{T_X + T_Y}$$

where all the bundles above are assumed to have been restricted to W.

3.1. Theorem. *The following statements are equivalent.*

(1) There exists an isomorphism of dg functors $\mathbf{D}(X) \to \mathbf{D}(Y)$

$$j^* i_*(-) \cong q_*(p^*(-) \otimes \mathbb{S}(E^\vee[1])).$$

(2) W' *is isomorphic to* $\mathrm{Tot}_W E[-1]$ *as dg schemes over* $X \times Y$.

(3) W' *is formal as a dg scheme over* $X \times Y$.

(4) The inclusion $W \to W'$ *splits over* $X \times Y$.

(5) The inclusion $W \to W'$ *splits to first order over* $X \times Y$.

(6) *The short exact sequence*

$$0 \to T_X + T_Y \to T_S \to E \to 0$$

of vector bundles on W *splits.*

Proof. We shall prove the following chain of implications and equivalences

$$(3) \Leftrightarrow (2) \Rightarrow (4) \Rightarrow (5) \Leftrightarrow (6) \Rightarrow (2) \Rightarrow (1) \Rightarrow (6).$$

The main statements that require proof are the implications $(1) \Rightarrow (6)$ and $(6) \Rightarrow (2)$. Before launching into discussing these we briefly explain the remaining implication above.

The equivalence of statements (2) and (3) is evident from the definition of formality and the calculation of the cohomology sheaves of $\mathscr{O}_{W'}$ from (1.3). The implications $(2) \Rightarrow (4) \Rightarrow (5)$ are trivial. The equivalence $(5) \Leftrightarrow (6)$ is a dg version of [**?**, 20.5.12 (iv)]. The implication $(2) \Rightarrow (1)$ follows from the considerations in (2.5).

For the remainder of the proof we replace the initial intersection problem with the problem of intersecting $X \times Y$ with the diagonal in $S \times S$, as in (2.6). We keep denoting the new spaces and embeddings by X, Y, and S, i, j, etc. Thus the new S is the old $S \times S$, the new X is the diagonal in the old $S \times S$, and the new Y is the old $X \times Y$.

We reformulate (1), (2), and (6) of the theorem in the new setting. Statements (1) and (2) become the statements that there exist isomorphisms

$$j^* i_* \mathscr{O}_X \cong q_* \mathbb{S}(E^\vee[1])$$

as objects of $\mathbf{D}(X \times Y)$ and as commutative dg algebra objects in $\mathbf{D}(X \times Y)$, respectively. The short exact sequence of (6) becomes the sequence

$$0 \to N_{W/Y} \to N_{X/S}|_W \to E \to 0.$$

Assume that (1) holds, in other words that there is an isomorphism $j^* i_* \mathscr{O}_X \cong q_* \mathbb{S}(E^\vee[1])$. We compute $\mathscr{H}^{-1}(q^* j^* i_* \mathscr{O}_X)$ in two different ways. On one hand we have

$$\mathscr{H}^{-1}(q^* j^* i_* \mathscr{O}_X) = \mathscr{H}^{-1}(q^* q_* \mathbb{S}(E^\vee[1])) = E^\vee \oplus N_{W/Y}^\vee$$

by a calculation analogous to the one in (1.3) and the assumption of (1). On the other hand a similar calculation for the map i shows that

$$\mathscr{H}^{-1}(q^* j^* i_* \mathscr{O}_X) = \mathscr{H}^{-1}(p^* i^* i_* \mathscr{O}_X) = p^* N_{X/S}^\vee.$$

We conclude that

$$N^\vee_{X/S}|_W \cong E^\vee \oplus N^\vee_{W/Y},$$

which using Lemma 3.2 below shows that the sequence

$$0 \to N_{W/Y} \to N_{X/S}|_W \to E \to 0$$

is split, thus proving (6). We have proved (1) $\Rightarrow$ (6).

We turn our attention to the implication (6) $\Rightarrow$ (2). Assuming that (6) holds, fix an isomorphism $N_{X/S}|_W \cong N_{W/Y} \oplus E$. Consider the map

$$q^*j^*i_*\mathscr{O}_X \cong p^*i^*i_*\mathscr{O}_X \cong \mathbb{S}(N^\vee_{X/S}|_W[1]) \cong \mathbb{S}(E^\vee[1]) \otimes \mathbb{S}(N^\vee_{W/Y}[1]) \to \mathbb{S}(E^\vee[1]).$$

Here the first isomorphism comes from the equality $q^*j^* \cong p^*i^*$, the second isomorphism is the main result of [**?**], the third one arises from the isomorphism

$$N_{X/S}|_W \cong N_{W/Y} \oplus E,$$

and the last map uses the projection

$$\mathbb{S}(N^\vee_{W/Y}[1]) \to \mathscr{O}_W.$$

Using the adjunction $q^* \dashv q_*$ the above map on W gives rise to a map on Y

$$j^*i_*\mathscr{O}_X \to q_*\mathbb{S}(E^\vee[1]).$$

Since all the maps involved are dg algebra maps, all that is left to prove (2) is to check that this last map is an isomorphism in $\mathbf{D}(Y)$. This is a local statement, which can be checked (locally) using Koszul resolutions. This concludes the proof. □

3.2. Lemma. *Let* k *be a field, and consider a short exact sequence*

$$0 \to M' \to M \to M'' \to 0$$

in a fixed k*-linear abelian category. Assume that* $\mathsf{Hom}(M', M')$, $\mathsf{Hom}(M, M')$, *and* $\mathsf{Hom}(M'', M')$ *are finite dimensional over* k. *If* M *is abstractly isomorphic to* $M' \oplus M''$, *then the short exact sequence splits.*

Proof. Consider the long exact sequence

$$0 \to \mathsf{Hom}(M'', M') \to \mathsf{Hom}(M, M') \to \mathsf{Hom}(M', M') \to \mathsf{Ext}^1(M'', M').$$

By the assumption that M is isomorphic to $M' \oplus M''$ we know that $\mathsf{Hom}(M, M')$ is abstractly isomorphic to $\mathsf{Hom}(M', M') \oplus \mathsf{Hom}(M'', M')$. Counting dimensions shows that the map

$$\mathsf{Hom}(M, M') \to \mathsf{Hom}(M', M')$$

is surjective, hence the map

$$\mathsf{Hom}(M', M') \to \mathsf{Ext}^1(M", M')$$

is zero. Therefore the class of the given extension, which is the image of the identity in $\mathsf{Hom}(M', M')$ under the above map, is zero.

4. Applications to orbifolds

In this section we discuss how our main result Theorem 1.8 can be used to understand the structure of derived fixed loci and of the loop space of an orbifold.

4.1. We review the setup in (1.10). Let Z be a smooth variety over a field k, and let φ be an automorphism of Z of finite order n. Let W be the fixed locus of φ. We shall assume that the characteristic of k is either zero or greater than $\max(n, \mathsf{codim}_Z W)$.

Note that the ordinary fixed locus W can be understood as an intersection,

$$W = \Delta \times_{Z \times Z} \Delta^{\varphi},$$

where Δ and Δ^{φ} denote the diagonal in $Z \times Z$ and the graph of φ, respectively. As such the expected dimension of W is zero. Whenever $n > 0$ it is important to understand the failure of this intersection problem to be transversal, by studying the derived intersection space

$$W' = \Delta \times Z \times Z^R \Delta^{\varphi}.$$

We shall sometimes call this space the derived fixed locus of φ.

4.2. Theorem 1.8 shows that in order to understand the structure of W' we need to study the short exact sequence

$$0 \to T_{\Delta} + T_{\Delta^{\varphi}} \to T_{Z \times Z} \to E \to 0$$

where E is the excess bundle for this intersection problem. We shall prove that this sequence is always split, under the assumptions we made for the characteristic of k. We begin with a lemma.

4.3. Lemma. *In the setup of Theorem 1.8, assume that the map* $X \to S$ *is split to first order, and that the short exact sequence*

$$0 \to N_{W/Y} \to N_{X/S}|_W \to E \to 0$$

splits. Then the six equivalent statements of Theorem 1.8 are all true.

Proof. It is easy to see that the two conditions of the lemma imply that the short exact sequence of (6) of Theorem 1.8 splits. Equivalently, these two conditions are what was used in the proof of Theorem 1.8 after changing the problem to an intersection of the diagonal with $X \times Y$.

4.4. Theorem. *Assume we are in the setup of (4.1). Denote by* $(T_Z)_\varphi$ *the bundle on* W *of coinvariants of the action of* φ *on* T_Z*,*

$$(T_Z)_\varphi = \frac{T_Z}{\langle v - \varphi(v) \rangle}.$$

Then the derived fixed locus W' *is isomorphic, as a dg scheme over* $Z \times Z$*, to the total space over* W *of the dg vector bundle* $(T_Z)_\varphi[-1]$*,*

$$W' \cong \mathrm{Tot}_W\left((T_Z)_\varphi[-1]\right).$$