1 Counting for Cavemen

In the first part of the course we will discuss ways in which we communicate numbers. What symbols do we choose and how do we use them? In particular, we will consider **sign-value systems** such as the Roman Numeral system, and **positional number systems**, such as the Hindu-Arabic decimal system; we will also consider alternate positional number systems, such as the binary one. Ultimately, the abstract notion of a number is different from the symbols we use to communicate it. But as communication of numerical information plays such an important part of our everyday lives, finding "good" ways of representing numbers is a critical first step in our attempt to understand them.

1.1 Abstracting Numbers

We begin this course by considering numbers and the symbols that we use to represent them. As we will soon learn, the symbols which we use to represent numbers are totally meaningless. But they're good to know, since they have more or less become the convention by which people in the world around us denote numbers. We will discuss several ways of communicating numbers in writing. Most of you are familiar with the one method (sign-symbol), everyone is familiar with a second method (decimal), and some of you are likely familiar with the second and half-way (binary).

Since the time that you were two or three years old, you've been able to understand certain abstract properties. For example, a two-year old can look at the shapes on the left of Figure 1.1 and understand that they share something in common, different from those shapes in the middle or on the right. They can somehow abstract the properties of "triangle"-ness, "circle"-ness, or even "rectangularity", even if they don't understand those English words. A two-year old can also look at several of these shapes and understand that some of them have the same color as others. This is true even if they can't articulate the words green, red, or yellow. It might be difficult to carefully define some these properties, but even a two- or three-year old child can understand these abstract concepts.

Quantity is another important abstract concept that you develop at a very young age. A three-year old knows that there are the same number of each kind of shape, or the same number of shapes with each color. They might or might not know what this symbols '3' or '4' mean, but they know that there are the same number of squares as triangles as rectangles, or the same number of blue shapes as there are red shapes. It is important to understand that the concept of numbers is distinct from the symbols that we might use to represent them.

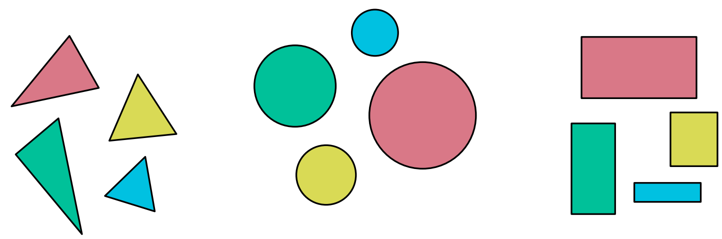
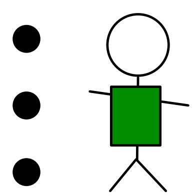


Figure 1: Several triangles, circles, and rectangles of different sizes and colors.

1.2 Communicating Numbers

Once humans started writing and reading thousands of years ago, and not just drawing pictures, it became necessary to develop a way in which to communicate numbers. We are dealing with very early times, long before the invention of the symbols 1, 2, 3, etc. that we use today to write down numbers.

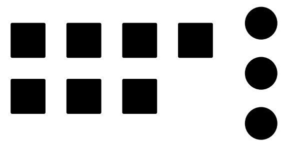
We begin now to carefully consider the development of methods by which to communicate numbers. What would be a very simple way to write down a number? Perhaps we can put down one dot for 1, two dots for 2, three dots for 3, and so forth. If I wanted to communicate that there are three people wearing green shirts, perhaps I would draw the figures below. Using one dot



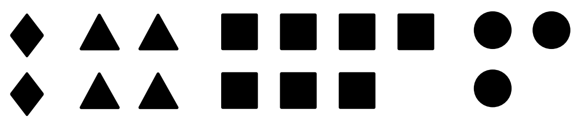
to represent each person with a green shirt is fairly intuitive and easy to draw. Such a method might be particularly convenient to implement on, for example, a piece of wood or the wall of a cave. However, even if using dots for each person is very intuitive, certain problems can arise when considering larger quantities. Let's imagine, for example, that we wanted to communicate that there are 73 people sitting in this room. We would then need to draw 73 dots on the board, one for each person. Not only would drawing so many dots take a very long time, reading them would also be difficult. It would take much time to count all 73, and also to be sure that we didn't make a mistake in counting so many.

1.3 Sign-Value Systems

To help remedy the problem of large quantities, we might consider introducing an additional symbol, perhaps to represent groups of 10. We could use a square, for example, for each group of ten. To represent 73 in symbols, we would then write down seven squares and three dots, as below. Adding the square can help



to some extent, but it too will fail us when attempting to communicate even larger numbers. To remedy this situation we can introduce new symbols that represent increasingly larger orders of magnitude. We could use a triangle to represent groups of 100, and a diamond to represent groups of 1000. The figure below then shows how we could represent the number 2473 using these symbols. This kind of system which uses distinct symbols to represent different quantities



is know as a symbol-value, or sign-value, system, and is one of the earliest types of numbers systems used.

Roman Numerals

One of the best-known symbol-value systems is the Roman numeral system, familiar to most students taking this class. If you have ever read a book, then you may have noticed that page numbers of the introduction or preface are often written using lower-case Roman numerals. Roman numerals appear in many others are of everyday life, though they are rarely used today in mathematics.

In the Roman numeral system, each of 7 letters from the Latin alphabet represent a different number. The value indicated by several different symbols

I	V	X	L	C	D	M
1	5	10	50	100	500	1000

put together is the sum of the values of each symbol. The number 170 would then be written CLXX, the number 2015 would then be written MMXV, and so forth. Converting in the other direction is also relatively straightforward. We briefly discussed additive and subtractive forms of Roman numerals.

Operations with Sign-Value Systems

Adding two numbers represented in Roman numerals is quite simple – we can just put together all of the symbols from the two numbers into one group. For example XXVI + XXXVI would just equal XXVIXXXVI. It might then be helpful to re-order them, such as from greatest values to least value; in this case we would have XXXXXVVII. Finally, we could combine symbols in the standard way, so that five X's are converted to an L, and two V's are converted into an X. We would then end up with LXII.

Multiplication can also be done quite directly in several ways. One way could work by making a multiplication table that is similar to the one we use for multiplication in the standard decimal system. For example:

	I	**V**	**X**	**L**	**C**	**D**	**M**
I	I	V	X	L	C	D	M
V	V	XXV	L	CCL	D	MMD	–
X	X	L	C	D	M	–	–
L	L	CCL	D	MMD	–	–	–
C	C	D	M	–	–	–	–

Here we placed a hyphen '–' instead of numbers which would take much space to write out.

To use this table to multiply two numbers, such as XVI and VII, we could go through each symbol in one (let's say X, V, and I) and multiply it by each symbol in the other (V, I, and I). We would use the multiplication to know the result of multiplying each pair of symbols. After obtaining all products (L,X,X, and XXV,V,V, and V,I,I), we could combine those products and order the symbols to give us LXXXXVVVVII, which we could then reduce to CXII. Those of us familiar with the decimal system, and how to convert between the two systems, can confirm that 16×7 is indeed 112.

Subtraction is a bit trickier but can also be done quite directly. As an exercise, the reader may think about how division might be done effectively. Division appears to be the most difficult operation to do with numbers represented using Roman numerals.

Limitations

Sign-value systems such as Roman numerals have long-ago fell into disuse for general mathematical purposes. We considered several limitations of such systems. First, there is no obvious way to represent the number zero. We could, of course, leave some space blank or else write out a word with the meaning of none, but it is often convenient to have a special symbol to indicate zero.

Another limitation might be the inability, at least using the classical Roman numeral system, to represent negative numbers. This, however, does not seem like a significant limitation, as we can introduce the same symbol '$-$' that we use in our familiar decimal system. For example, we could use $-$XVII to represent the number -17.

We have mentioned previously that division using Roman numerals might be particularly complicated. In a similar vein, communicating and manipulating fractions can be somewhat difficult using sign-value systems. This problem can be solved by introducing a new symbol to represent a value such as 1/2 or 1/10 (or 1/12). For example, if we used the symbol $\odot$ to represent 1/2 and $\bullet$ to represent 1/10, then we could represent 0.3 as $\bullet\bullet\bullet$ and 0.6 by $\odot\bullet$. We could introduce new symbols to represent smaller numbers such as 1/50 and 1/100.

Each of the previous problems can be solved, to some degree, with some clever "upgrading" of classical sign-value systems. One problem, however, will likely not go away. In sign-value systems, we need more and more symbols for discussing ever-more complicated numbers. In particular, as we attempt to represent larger and larger numbers (or smaller and smaller numbers), we need to introduce more and more symbols. For example, we earlier introduced symbols that allowed us to easily represent numbers from 1 up to 1000. If we wanted to, however, represent numbers in the tens or hundreds of thousands, and certainly in the millions, we would need to introduce yet new symbols for this purpose. If we don't introduce new symbols, we will need to draw at least one thousand symbols just to represent the number 1,000,000. The regular need for new symbols may have been one of the primary reason for the fall of sign-value systems from common use.

1.4 Positional Number Systems

In symbol-value systems, the position of a symbol does not affect its value. For example, the symbol 'X' that appears in these three numbers: 'CLX', 'CXI', and 'XVI', always indicates 10, even though it appears in three different positions in the three strings of symbols. We will now begin to discuss the more ubiquitous **positional number** systems. In such systems, the value of a number depends on its position. For example, the symbol '3' can either indicate three, as in '13', '193', and '3', or else it could indicate thirty, three hundred, or three thousand, as in '1230', '1350', and '3262'. It is the *position* of the symbol, in the string of symbols, that tells us how much to add to our total. This is the system that all of us grew up with and with which we are all intimately familiar.

Decimal System

Our traditional decimal system makes use of ten distinct symbols: '0', '1', '2', '3', '4', '5', '6', '7', '8', and '9'. I have specifically placed the symbols between apostrophes to highlight that we are not discussing the numbers between zero and nine, but the actual symbols that we commonly use to represent them in writing. Of course, these symbols themselves are not of any particular importance, and we could just as well have used the ten symbols ')', '!', '@', '#', '$', '%', '^', '&', '*', and '('. In any case, the symbols we use allow us to represent not only the numbers between zero and nine, but actually arbitrarily larger or small numbers too.

Although the following might be obvious, let us briefly consider some simple examples of numbers represented in the decimal (positional) system. Let's consider the number 4153. This communicates that we have 4 groups of a thousand, 1 group of a hundred, 5 groups of ten, and 3 ones. We can think of this as:

$$4153 = 4 \times 1000 + 1 \times 100 + 5 \times 10 + 3 \times 1. \tag{1}$$

Writing it this way can help highlight what is really going on. We can rewrite this equation making use of the fact that we are using a system with 10 symbols:

$$4153 = 4 \times 10^3 + 1 \times 10^2 + 5 \times 10^1 + 3 \times 10^0. \tag{2}$$

Each symbol in our string of symbols indicates how many groups of a certain size we have. The sizes of those groups are powers of 10. In particular, we have groups of 1, 10, 100, 1000, and so forth. This may seem pedantic, but it highlights the role which the number 10 plays in determining the values represented by each symbol.

Other Bases

Although the decimal system, with its 10 symbols, is most familiar to us, there is no reason we can't use other numbers of symbols instead. Ten is no more special than 4 or 7 or 13. The number of symbols used in a particular positional number system is called its **base**. The decimal system is a base-10 system since it uses 10 symbols. In this section we will consider using a base-7 system, using seven symbols to represent numbers. Let us use the following seven symbols: '0', '1', '2', '3', '4', '5', and '6'. It is clear that if one of these symbols is reserved to indicate zero, then we can count to at most six with these symbols, if we restrict ourselves to using only one symbol one time.

Let us begin counting: 0, 1, 2, 3, 4, 5, 6. How can we represent seven? Our instinct might push us to use the symbol '7', but we must remember that there is no such symbol in our base-7 system. Instead, we might indicate that we have exactly one group of 7. We can't write just 1, since that would indicate one, but we can write 10, to indicate that we have one group of seven, and nothing else. The next number, what we call eight, can be written 11, which would indicate one group of 7's, and one 1's. We can continue counting 12, 13, 14, 15, 16. The last number indicates thirteen, since it means we have one group of 7's and six 1's, which add together to thirteen. The next number, fourteen, should be represented as 20, which means two groups of 7's, and zero 1's. Using only two characters, the largest number we can represent is forty eight, which we write 66, which means six groups of 7's, and six 1's, which of course adds up to forty eight. The next number we want to represent is forty-nine, which we write as 100, which means one group of 49 (7^2) and zero groups of 7's (7^1) and zero 1's (7^0). That is, each symbol represents some group of 7^n, where n is an integer equal to or larger than 0.

In order to avoid ambiguities, we might write a subscript to indicate the base being used. For example, we can write:

$$100_7 = 49_{10} \tag{3}$$

which states that forty-nine equals forty-nine. Of course, the number is written in a different way on the left side than it is on the left side, but it is still the same number. When there is little room for ambiguity we sometimes leave off the subscript.

Let us consider a larger number 4153_7. What does this string of symbols

'4153' mean to us, when understood to represent a number in base-seven?

$$\begin{aligned} 4153 &= 4 \times 7^3 + 1 \times 7^2 + 5 \times 7^1 + 3 \times 7^0 && (4) \\ &= 4 \times 343 + 1 \times 49 + 5 \times 7 + 3 \times 1 && (5) \\ &= 1372 + 49 + 35 + 3 && (6) \\ &= 1459. && (7) \end{aligned}$$

All numbers on the right-hand side should be understood as using the standard base-10 system. Therefore we have $4153_7 = 1459_{10}$.

How would we represent $7 \times 7 = 49$ in the base-7 system? Each 7_{10} would be represented as 10 in the base-7 system, and their product, 49, is represented as 100. So in base-7, we will have $10 \times 10 = 100$. Although this looks like a very familiar equation to us, it is worth keeping in mind that we are now working in base-7. Actually, you should be able now to see for yourself that in any positional system, regardless of the base, we will always have $10 \times 10 = 100$. This equation will be as true in base-2 as it is in base-7, base-10, or base-23. One way to see this is that the second position from the right, always indicates the number of groups of b^1, where b is the base of the system. Therefore 10 always represents exactly b^1. Multiplying these together of course gives us b^2, which is always indicated by the third digit from the right. If this isn't clear to you, spend some time thinking about it. If you understand this point, then you understand a good deal about positional numbers systems.

Binary System

What is the fewest number of symbols we can get away with using? In a positional number system that always uses one symbol for zero, we must have at least two symbols, one for zero, and one for more than that. A system with exactly two symbols is called a **binary** number system, and such number systems form the basis of all modern electronics. When writing in English text, the two symbols 0 and 1 are traditionally used, but of course, the particular symbols we use are not important, and we could just as well use ⌑ and $\Diamond$.

Let us start counting, as we had done before. We can use 0 to indicate zero, and 1 to indicate one. Now we are stuck, as we have run out of symbols! What do we do? We write 10, since the symbol on the left represents the number of groups of $2^1 = 2$'s we have, and the symbol on the right tells us how many 1's to add. If we want to represent three, then we should write 11, which indicates that we have 1 group of 2's and 1 group of 1's. Each position, represents a group of 2^n, where n indexes, from 0 upwards, the position of the digit from the right.

Let us consider an example:

$$\begin{aligned} 10110 &= 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 && (8) \\ &= 1 \times 16 + 0 \times 8 + 1 \times 4 + 1 \times 2 + 0 \times 1 && (9) \\ &= 16 + 0 + 4 + 2 + 0 && (10) \\ &= 22. && (11) \end{aligned}$$

As before, the numbers on the right are written in base-10. We can summarize these equations by writing $10110_2 = 22_{10}$. The right-most digit represents 1's, the second right-most represents 2's, then 4's, 8's, 16's, etc. We can then write out all numbers using only two symbols. Although we need to use more characters in binary to represent the same number as we do in a decimal system (we needed five instead of only two), we can get away with using only two different symbols.

We considered how we do addition in different positional number bases. In particular, we add numbers in columns in the same way. When we go "over the limit" (for example, if we have more than 9 in base 10, or more than 6 in base 7), then we "carry a 1" to the next column, and leave only the remainder after having "moved over" these numbers.

Hexadecimal System

After the binary numbers system, the second-most-widely used number system in technology is the hexadecimal system, which uses 16 symbols. Conventionally, these symbols are represented by our regular numerals as well as the first six letters of the English alphabet. Each symbols represents a number between 0

0	1	2	3
4	5	6	7
8	9	A	B
C	D	E	F

and 15. 'A' is a symbol that indicates ten, 'B' denotes eleven, ..., and 'F' indicates 15. The English letters are sometimes represented in upper-case and sometimes in lower-case; there is no uniformity about this convention.

If you have spent some time internalizing how positional number systems work, then this example should make sense. Let us do just one example of representing a number in this system.

$$\begin{aligned} 2A8F &= 2 \times 16^3 + 10 \times 16^2 + 8 \times 16^1 + 15 \times 16^0 && (12) \\ &= 8192 + 2560 + 128 + 15 && (13) \\ &= 10895. && (14) \end{aligned}$$

1.5 Conclusions

The important takeaway from the first part of the course is this: numbers are distinct from the way in which we represent them. Consequently, no mathematical property of a number depends on how we write it. The number 13 is odd whether we write it as thirteen little dots, 'XIII' (in Roman numerals), '16' (in base-7), '1101' (in binary), or 'D' (in hexadecimal). It is still prime and it still greater than ten and smaller than twenty. Properties of numbers do not depend on the way in which we write them. Number systems have developed to help us communicate with one another about quantity, a property of fundamental importance.