5.2 Graph Isomorphism

Most properties of a graph do not depend on the particular names of the vertices. For example, although graphs A and B is Figure 10 are technically different (as their vertex sets are distinct), in some very important sense they are the "same"

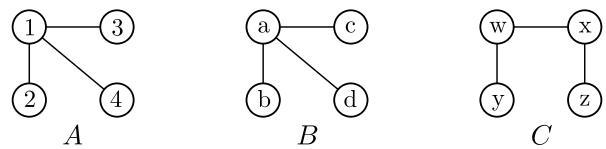

Figure 10: Two isomorphic graphs A and B and a non-isomorphic graph C; each have four vertices and three edges.

graph. For example, both graphs are connected, have four vertices and three edges. However, notice that graph C also has four vertices and three edges, and yet as a graph it seems different from the first two. Isomorphism is the idea that captures the kind of sameness that we recognize between A and B, and which distinguishes both of them from C.

Definition 19. *Two graphs G_1 and G_2 are* **isomorphic** *if there exists a matching between their vertices so that two vertices are connected by an edge in G_1 if and only if corresponding vertices are connected by an edge in G_2.*

In Figure 10, we can match the vertices of graph A with those of graph B in such a way: 1 is matched with a, 2 with b, 3 with c, and 4 with d. An edge connects 1 and 3 in the first graph, and so an edge connects a and c in the second graph. Likewise, no edge connects 3 and 4 in the first graph, and so no edge connects c and d in the second graph. Regarding the two graphs in Figure 10, we can write $A \cong B$ to denote this isomorphism. Although we matched vertices of A with those of B in one particular way, there could be several ways to do. For example, we could match 1 with a, 2 with c, 3 with d, and 4 with b; there are several other ways to do this. We often use the symbol $\cong$ to denote isomorphism between two graphs, and so would write $A \cong B$ to indicate that A and B are isomorphic.

Although graphs A and B are isomorphic, i.e., we can match their vertices in a particular way, graph C is not isomorphic to either of A or B. As hard as we try, we will fail to find a matching between vertices of A, for example, and those of C that maintain edge-connections between corresponding vertices. We can write $A \ncong C$ to indicate that A and C are not isomorphic.

Graph theorists are primarily interested in properties of graphs that do not change when vertices are relabeled; sometimes they will discuss "properties that are invariant under isomorphisms" which conveys this idea. We note that if $G_1 \cong G_2$, then many properties of G_1 and G_2 must be the same. For example, the number of vertices and edges in the two graphs must be identical. G_1 is *connected* if and only G_2 is connected. G_1 is k-regular if and only if G_2 is k-regular. G_1 is bipartite if and only if G_2 is bipartite. The numbers of vertices with degree 0, 1, 2, etc. must be identical. None of the properties listed here change when vertices are relabeled. Conversely, if two graphs G_1 and G_2 differ with respect to any of these properties, then we can know that G_1 and G_2 are not isomorphic.

Many properties of individual vertices also do not change "under isomorphisms", or relabeling of the vertices. For example, if vertex u in graph G_1 can be matched with vertex v in G_2, then we must have $\deg(u) = \deg(v)$. If $\deg(u) \neq \deg(v)$, then we cannot match up the two vertices.

Determining whether two graphs are isomorphic is not always an easy task. For graphs with only several vertices and edges, we can often look at the graph visually to help us make this determination. In the following pages we provide several examples in which we consider whether two graphs are isomorphic or not. Our focus here is more on visual presentations of graphs, but we could also consider presentations of graphs in terms of sets.

Example 1

A relabeling of vertices of a graph is isomorphic to the graph itself. Consider the three isomorphic graphs illustrated in Figure 11. The first two graphs illustrate

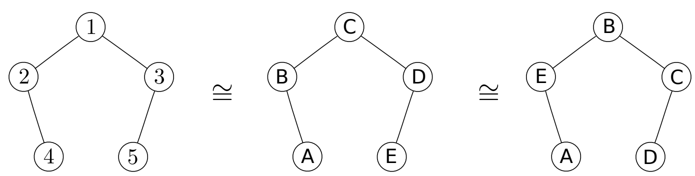

Figure 11: Three isomorphic graphs.

a change of using letters to using numbers to label the graphs. The second pair of graphs are also isomorphic as only the labels were changed. We can match vertices in the second graph with those in the third graph to satisfy the isomorphism requirements. Another way to think about graph isomorphism is by removing all vertex labels from two graphs. It is clear for these examples that all three graphs are then identical.

Example 2

Like relabeling, moving around vertices also does not change important graph properties. The two graphs illustrated below are isomorphic since edges connected in one are also connected in the other. In fact, not only are the graphs isomorphic to one another, but they are in fact identical. Notice that each vertex in one graph is matched to itself in the other graph.

Example 3

The figure below illustrates another pair of isomorphic graphs. Although the graphs have a slightly different shapes from one another, we can still find a

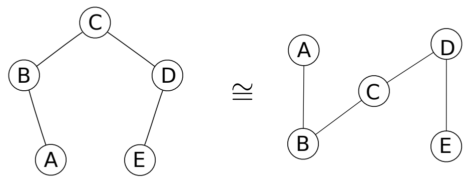

Figure 12: Two isomorphic graphs.

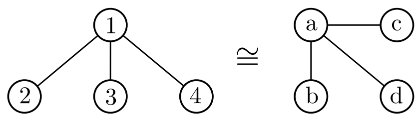

Figure 13: Two isomorphic graphs.

1-1 matching between the vertices so that if pairs of vertices are connected by an edge in one graph, then corresponding vertices will be connected in the other. The same matching given above (a1, b2, c3, d4) will still work here, even though we have moved the vertices around. It is worth noting that several other matchings would also work. For example a1, b3, c4, d2 would also be a good matching. In fact, in this example, as long as a is matched with 1, then b, c, and d can be matched in any order with 2, 3, and 4.

Example 4

Any two cliques K_n and K_m are isomorphic if any only if $n = m$. If $n \neq m$, then it is clear that we cannot have a 1-1 matching between the vertices of the two graphs, because there will be more vertices in one graph than in the other. If $n = m$ then any matching will work, since all pairs of distinct vertices are connected by an edge in both graphs. Notice that in the graphs below, any matching of the vertices will ensure the isomorphism definition is satisfied.

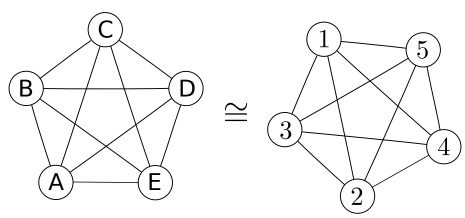

Figure 14: Two complete graphs on five vertices; they are isomorphic.

Example 5

Just because two graphs have the same number of vertices and edges does not mean that they are isomorphic. In fact, even if the degrees of all vertices are identical, still the two graphs can be non-isomorphic. Although $G_1 \cong G_2$ (we

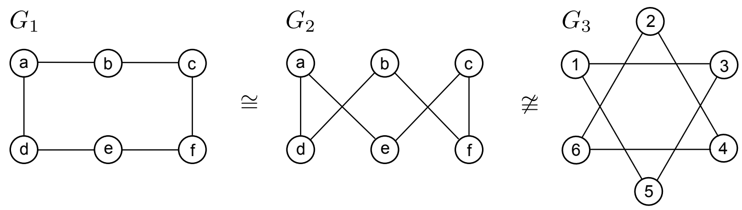

Figure 15: Three 2-regular graphs on six vertices; the first two are isomorphic; the third one is not.

can imagine deforming either into the other), G_2 and G_3 are not isomorphic. No matter how we relabel the vertices of G_2, it will remain a connected graph. Likewise, no matter how we relabel the vertices of a G_3, it will remain unconnected. Two graphs that are isomorphic must both be connected or both disconnected.

Example 6

Below are two complete graphs, or cliques, as every vertex in each graph is connected to every other vertex in that graph. As a special case of Example 4,

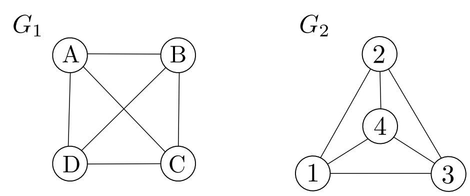

Figure 16: Two complete graphs on four vertices; they are isomorphic.

we already know that these two graphs are isomorphic since they have the same number of vertices. The two drawing here, however, highlight a particularly interesting feature of certain graphs. We have already seen that we can generally draw graphs in many different ways without changing their overall structure. So long as we don't disconnect any vertices that are connected to each other, and so long as we don't attach any vertices that were previously disconnected, we are free to move the vertices and edges as we please. In this example, in the first way we drew the graph, two of its edges (AC and BD) crossed one another. However, in the second drawing, of an isomorphic graph, we were able to draw the graph in such a way that no two edges cross. In other words, there are some graphs that can be drawn without edges crossing. Can *all* graphs be drawn in such a way that no two edges cross? This question leads us to consider the next big topic in graph theory, the study of planarity.