5.3 Planar Graphs and Euler's Formula

Among the most ubiquitous graphs that arise in applications are those that can be drawn in the plane without edges crossing. For example, let's revisit the example considered in Section 5.1 of the New York City subway system. We considered a graph in which vertices represent subway stops and edges represent direct train routes from one subway stop to the next. We might wonder whether such a graph can be drawn without any edges crossing. Why should we care about this? Since each edge represents a subway line, the crossing of edges represents two subway lines that cross paths. For that to happen, either train tracks will need to cross, an engineering feat involving careful consideration of potentially-conflicting schedules and the engineering of special tracks, or else subway lines must be built at different depths below ground. Both of these options are costly and challenging. It turns out that the graph of the NYC subway system cannot be drawn in such a way, and indeed many subway lines run at different depths below ground in various parts of the city.

In this section we consider which graphs can be drawn on paper without edges crossing and which graphs cannot.

Definition 20. *A graph G is* **planar** *if it can be drawn in the plane in such a way that no pair of edges cross.*

Attention should be paid to this definition, and in particular to the word 'can'. Whether or not a graph is planar does *not* depend on how it is actually drawn. Instead, planarity depends only on whether it 'can' be drawn in such a way. By defining this property in this more abstract way, we can ensure that planarity is preserved under isomorphisms. If planarity depended on how a particular graph was drawn, then we could have two isomorphic graphs, such that one is planar and the other is not. Furthermore, graphs that are only described abstractly through a vertex set V and an edge set E, and without being drawn, could not be described as planar or not, since there could be multiple ways of drawing it.

Determination of Planarity

Sometimes it is easy to see that a particular graph is planar, especially when it is drawn in such a way. In Examples 1, 2, and 3 of Section 5.2, we can readily see that all graphs are planar, as no edges cross. However, we have noted in the discussion of Example 6 that it is sometimes difficult to determine that a particular graph is planar just from looking at it. For example, G_1 in Example 6 of Section 5.2 might give the mistaken impression that K_4 is a non-planar graph, even though G_2 there makes clear that it is indeed planar; the two graphs are isomorphic. These observations motivate the question of whether there exists a way of looking at a graph and determining whether it is planar or not.

Euler's Formula for Planar Graphs

The most important formula for studying planar graphs is undoubtedly Euler's formula, first proved by Leonhard Euler, an 18$^{\text{th}}$ century Swiss mathematician, widely considered among the greatest mathematicians that ever lived. Until now we have discussed vertices and edges of a graph, and the way in which these pieces might be connected to one another. In a sense, vertices are 0-dimensional pieces of a graph, and edges are 1-dimensional pieces. In planar graphs, we can also discuss 2-dimensional pieces, which we call faces. Faces of a planar graph are regions bounded by a set of edges and which contain no other vertex or edge.

Example 1

Several examples will help illustrate faces of planar graphs. The figure below

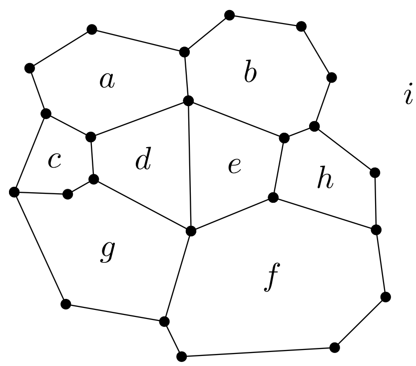

Figure 17: A planar graph with faces labeled using lower-case letters.

illustrates a planar graph with several bounded regions labeled a through h. These regions are called faces, and each is bounded by a set of vertices and edges. For reasons that will become clear later, we also count the region "outside" of the graph as a face; we sometimes call this the "outside" face.

Euler discovered a beautiful result about planar graphs that relates the number of vertices, edges, and faces. In what follows, we use $v = |V|$ to denote the number of vertices in a graph, $e = |E|$ to denote the number of edges in a graph, and f to denote its number of faces. Using these symbols, Euler's showed that for any connected planar graph, the following relationship holds:

$$v - e + f = 2. \tag{47}$$

In the graph above in Figure 17, $v = 23$, $e = 30$, and $f = 9$, if we remember to count the outside face. Indeed, we have $23 - 30 + 9 = 2$. This relationship holds for all connected planar graphs.

Example 2

An infinite set of planar graphs are those associated with polygons. The figure below illustrates several graphs associated with regular polyhedra. Of course,

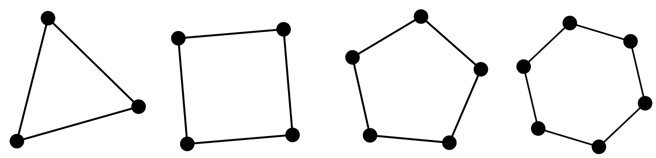

Figure 18: Regular polygonal graphs with 3, 4, 5, and 6 edges.

each graph contains the same number of edges as vertices, so $v - e + f = 2$ becomes merely $f = 2$, which is indeed the case. One face is "inside" the polygon, and the other is outside.

Example 3

A special type of graph that satisfies Euler's formula is a tree. A tree is a graph such that there is exactly one way to "travel" between any vertex to any other vertex. These graphs have no circular loops, and hence do not bound any faces. As there is only the one outside face in this graph, Euler's formula gives us

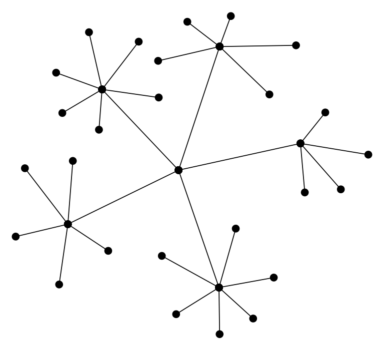

Figure 19: A tree graph – there are no faces except for the outside one.

$v - e + 1 = 2$, which simplifies to $v - e = 1$ or $e = v - 1$. Every tree satisfies this relationship and so always has one fewer edges than it has vertices.

Degree of a Face

In the same way that we were able to characterize a vertex by counting the number of edges adjacent to it, we can also characterize a face by its number of edges (or equivalently vertices) on its boundary. For example, consider Figure 20, which shows a graph we considered earlier. Here each face is labeled by its

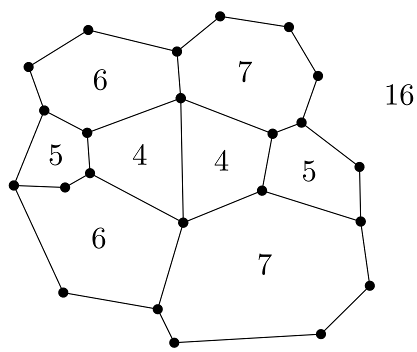

Figure 20: A planar graph with each face labeled by its degree.

number of edges. We use the word *degree* to refer to the number of edges of a face.

Definition 21. *The* **degree** *of a face f is the number of edges along its boundary. Alternatively, it is the number of vertices along its boundary. Alternatively, it is the number of other faces with which it shares an edge. The degree of a vertex f is oftentimes written* $\deg(f)$.

Every edge in a planar graph is shared by exactly two faces. This observation leads to the following theorem.

Theorem 8. *For all planar graphs, the sum of degrees over all faces is equal to twice the number of edges. In symbols, $\sum_i \deg(f_i) = 2|E|$, where f_i are the faces of the graph.*

This result might be seen as an analogue of a result we saw earlier involving the sum of degrees of all vertices (Theorem 7). These theorems help us understand the relationship between the number of edges in a graph and the vertices and faces of a (planar) graph.

Our definition of a graph (as a set V and a set E consisting of two-element subsets of V) requires that there be at most one edge connecting any two vertices. To understand this, consider two vertices a and b. We have learned before (in Section 2) that sets do not have duplicate elements. Therefore, the set of edges E can only contain the element $\{a, b\}$ one time. Such graphs are called *simple* and they have been the exclusive focus of our consideration in this section. Because our graphs are all simple, the smallest possible degree of a face is 3, since a face with degree two would require that two edges connect a pair of vertices.

We can use this result, along with Theorem 9, to obtain an important result about planar graphs. In particular, notice that since the degree of every face must be at least 3, we have $\sum_i 3 \leq \sum_i \deg(f_i)$. The left-hand side, however, evaluates to $3|F|$, where $|F|$ is the number of faces in the planar graph, because we are adding 3 for every face. Theorem 9 tells us that the right-hand side is equal to $2|E|$, where $|E|$ is the number of edges in the graph. Therefore, we can conclude:

Theorem 9. *For all planar graphs, $3|F| \leq 2|E|$, where $|F|$ is the number of faces and $|E|$ is the number of edges.*

This, of course, is equivalent to stating that $|E| \geq \frac{3}{2}|F|$; the number of faces of a planar graph ensures that we have at least a certain number of edges.

Non-planarity of K_5

We can use Euler's formula to prove that non-planarity of the complete graph (or clique) on 5 vertices, K_5, illustrated below. This graph has $v = 5$ vertices

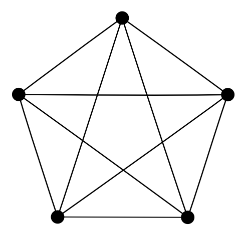

Figure 21: The complete graph on five vertices, K_5.

and $e = 10$ edges, so Euler's formula would indicate that it should have $f = 7$ faces. We have just seen that for any planar graph we have $e \geq \frac{3}{2}f$, and so in this particular case we must have at least $\frac{3}{2}7 = 10.5$ edges. However, K_5 only has 10 edges, which is of course less than 10.5, showing that K_5 cannot be a planar graph.

Faces in Non-planar Graphs

Non-planar graphs do not technically have faces – there does not seem to be any good way to discuss faces in cases when edges cross one another.