5 Graph Theory

Graph theory – the mathematical study of how collections of points can be connected – is used today to study problems in economics, physics, chemistry, sociology, linguistics, epidemiology, communication, and countless other fields. As complex networks play fundamental roles in financial markets, national security, the spread of disease, and other national and global issues, there is tremendous work being done in this very beautiful and evolving subject.

The first several sections covered number systems, sets, cardinality, rational and irrational numbers, prime and composites, and several other topics. We also started to learn about the role that definitions play in mathematics, and we have begun to see how mathematicians prove statements called theorems – we've even proven some ourselves. At this point we turn our attention to a beautiful topic in modern mathematics called graph theory. Although this area was first introduced in the 18th century, it did not mature into a field of its own until the last fifty or sixty years. Over that time, it has blossomed into one of the most exciting, and practical, areas of mathematical research.

Many readers will first associate the word 'graph' with the graph of a function, such as that drawn in Figure 4. Although the word graph is commonly

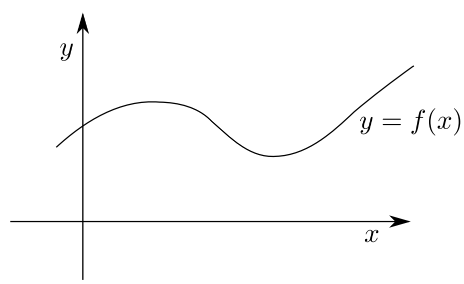


Figure 4: The graph of a function $y = f(x)$.

used in mathematics in this sense, it is also has a second, unrelated, meaning. Before providing a rigorous definition that we will use later, we begin with a very rough description and some examples.

The elementary pieces of graphs are quite simple: points and connections between them. Figure 5 shows several points and connections between some pairs of points. Each point can represent a person, a city, a webpage, or any other object. A connection between two points indicates some relationship between those vertices. For example, connections between two points might represent an existing flight route between two cities, a Facebook friendship between two people, or a pair of webpages that are linked to one another. Graph theory studies networks which can be described in this simplistic manner, as a set of points and connections between them.

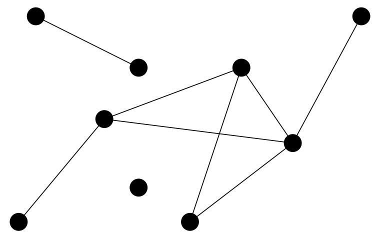


Figure 5: A graph, consisting of points and connections between them.

Example 1. We begin by considering several concrete examples to motivate our interest in studying graphs. Figure 6 shows a 1972 map of flight routes

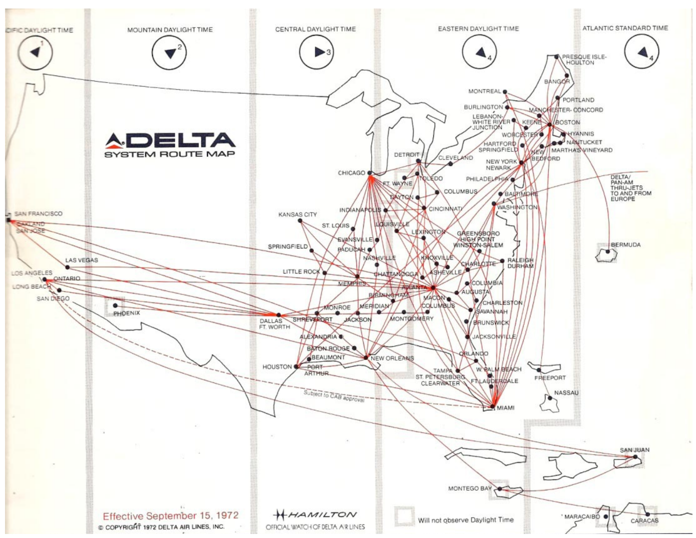


Figure 6: US cities connected by direct Delta Air Lines flights.

serviced by Delta Air Lines. Each black dot indicates a city or airport, and red curves indicates flight routes between pairs of cities. In studying such a map we might consider several very practical questions: Can a traveler leaving Lexington, Kentucky reach his or her destination in Los Angeles flying only Delta flights? If yes, what is the shortest route? What is the maximum number of flights one would need to take between any pair of cities in this map? The reader might notice that some cities are directly connected by flight routes to very few other cities, while other (such as Chicago, Atlanta, and Miami) are connected to many; cities directly connected to many other cities are oftentimes called hubs. One might wonder how adding extra flights between hubs will minimize the average number of connections necessary to travel between different cities. These, any many more questions, are routinely asked when studying maps of this kind.

Example 2. A second exceptionally interesting class of graphs are those found in social networks. In such a graph, each vertex might represent a person, and edges can represent pairs of people who are friends on Facebook. Such graphs raise many interesting questions studied by applied mathematicians and social scientists. For example, is there a good way to estimate the potential influence of various people in a social network? Naively, we might consider the number of friends each person has (or, equivalently, the number of edges incident with each vertex) in determining potential influence. However, the number of Facebook friends might be an inaccurate means of determining influence for the following reason. Consider Jack and Jill, two random Facebook users. Jack has 500 friends, but each of those friends has only 10 friends each. Jill has only 200 friends, but each of her friends has roughly 500 friends themselves. While in some sense Jack is more popular than Jill, he has many fewer friends of friends than does Jill. Of the two, Jill might be the more influential.

When considering the Facebook graph, we might also look for groups of people, each of whom is friends with everyone else in that group. In graph theory, such a subset of vertices is known as a clique. Much cutting-edge research in graph theory studies structural features of graphs, such as cliques.

One important difference between the two examples is the way in which they are drawn. In the first example, cities can be placed on a map in a location corresponding to its actual geographic location. There is a strong geometrical component to the information conveyed by this graph. In the second example, on the other hand, there is no clear way in which to draw the corresponding graph. People do not have unambiguous positions. Therefore, if we are to study a graph of a social network, we will only care about the "graph structure", that is the way in which points are connected, but ignore data regarding positions of the particular points.

Example 3. A third graph that is even more ubiquitous than social networks is that associated with the world-wide web itself. Imagine that we abstract each webpage on the internet as a vertex. An edge between vertex A and B might indicate the existence of some link on A that directs a web-surfer to B. Of course this graph is extremely large (some estimates place this number at over one trillion), and its structure is very complex. However, we might again ask about certain features of this graph. In the 1990's, two graduate students at Stanford developed a way of estimating the importance of a webpage by considering structural features of the graph of all webpages. These graduate students, Larry Page and Sergey Brin, ultimately converted their understanding of graphs into a very practical algorithm called PageRank, which has transformed Google into the largest and most successful search engine of the 21st century. This example introduced a new level of structure beyond what we have seen in the previous examples. In particular, in prior examples, relationships were symmetric in the sense that if A was connected to B, then B, of course, was connected to A. This is certainly the case in the Facebook network, as Jack cannot be friends with Jill if Jill is not also friends with Jack. Likewise, in the airline industry, if a traveler can fly from city A to city B on a given airline, it is generally the case that they can also fly from city B to city A. The world-wide web, however,

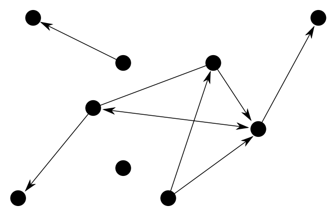


Figure 7: A directed graph, consisting of points and directed connections between them.

provides a more sophisticated kind of graph. It is certainly possible that webpage A links to B without webpage B linking to webpage A. The reader can certainly think for themselves of examples of networks that are non-symmetric. Graphs of such networks are called **directed graphs**. The graphs that we will consider in this class will all be **undirected graphs**.

5.1 Basics

We begin by describing some of the basics of graphs. Roughly speaking, a graph is a set of points and connections between those points; the points are called **vertices** and the connections are called **edges**. A more formal approach to defining a graph is given by the following:

Definition 13. *A* **graph** *G is an ordered pair of sets (V, E). Each element of E is a two-element subset of V. Each element of V is called a* **vertex** *and each element of E is called an* **edge***.*

The definition uses sets to define a new mathematical object called a graph. Keeping in mind our previous example, V can be a set of cities, people, or websites. Edges are two-element subsets of V indicating some relationship between those two elements.

Although sets can be thought of abstractly – as a pair of sets with certain properties – it is often convenient to think about graphs visually. For relatively small graphs, we can draw a point for each vertex, and can draw edges between vertices to indicate edges. Consider for example a graph given by $V = \{a, b, c, d\}$ and $E = \{\{a, b\}, \{a, c\}, \{a, d\}\}$. A representation of that graph is shown in Figure 8.

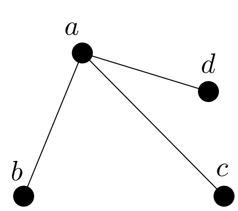


Figure 8: A graph consisting of four vertices and three edges.

Degree

One important property, perhaps the most important property, of a vertex v is the number of other vertices with which it is connected.

Definition 14. *The* **degree** *of a vertex v is the number of edges incident with v; equivalently, it is the number of elements of E of which v is itself an element. The degree of a vertex v is oftentimes written* $\deg(v)$.

In the example illustrated in Figure 8, the degree of a is 3, and the degree of b, c, and d is 1. Note that graphs can have vertices that are not connected to any other vertex; the degree of such vertices is 0. If a graph has no edges, then all of its vertices have degree 0. Note also that a graph with n vertices ($|V| = n$) can have vertices with degree at most $n - 1$, since any vertex can be connect to at most the other $n - 1$ vertices.

Relationship of Degrees to Edges

The degrees of the vertices give us one way of counting the number of edges in a graph. More specifically, the following is true for all graphs.

Theorem 7. *For all graphs, the sum of degrees over all vertices is equal to twice the number of edges. In symbols,* $\sum_i \deg(v_i) = 2|E|$, *where v_i are the vertices of the graph.*

Proof. If an edge is added between vertices u and v of a graph, then the degrees of u and v each increase by 1. Therefore, each edge increases the sum of all degrees by two. $\square$

Example 1

Some graphs have the property that all vertices are connected in some sense:

Definition 15. *A graph is* **connected** *if one can "travel" from any vertex to any other vertex along a series of edges. A graph that is not connected is called* **disconnected**.

If we think of a graph of the New York City subway system, in which vertices are subway stops and edges indicate direct trains from one subway stop to the next, then this graph is connected. It is always possible to reach any station from any other station in the system. On the other hand, if we consider the Facebook graph, in which vertices are people and edges indicate a pair of people that are friends, then such a graph is disconnected, as there are certainly Facebook users that have 0 friends. Note also that the graph pictured in Figure 5 is disconnected, while that pictured in Figure 8 is connected.

Example 2

Some graphs have the property that every vertex has the same degree. Such graphs are called regular:

Definition 16. *A graph is called* k**-regular** *if the degree of every vertex is* k*.*

Notice that a graph on n vertices can only be k-regular for certain values of k. First, of course k must be less than n, since the degree of any vertex is at most $n-1$. Furthermore, consider a graph with an odd number of vertices. If such a graph were k-regular for an odd value of k, then $\sum_i \deg(v_i)$ would be odd, which is not possible since it is equal to an even number $2|E|$. Therefore, it is not possible to have a k-regular graph on n vertices if both k and v are odd. Note that a regular graph need not be connected.

Example 3

A special type of regular graph is one in which any two vertices are connected by an edge.

Definition 17. *A* **complete graph** *is one such that for any two vertices* $u, v \in V$*, there exists an edge* $\{u, v\} \in E$*.*

A complete graph on n vertices is commonly denoted by K_n. Note that K_n is a $(n-1)$-regular graph. Also note that a complete graph on n vertices has $n(n-1)/2$ edges. We can see this by considering that the degree of each of the n vertices is $n-1$. Therefore, the sum of degrees of all vertices is $n(n-1)$, which according to Theorem 7 is equal to $2|E|$. Therefore, the number of edges is $|E| = n(n-1)/2$. Note also that all complete graphs are connected.

Example 4

Another important kind of graph is one in which the set of vertices and edges can be divided in a very particular way:

Definition 18. *A* **bipartite** *graph is one whose vertices* V *can be divided into two sets* A *and* B *such that there are no edges between any two vertices in* A *or between any two vertices in* B*.*

Notice that in Figure 9, all edges connect one vertex from group A to one vertex from group B; there are no edges between pairs of vertices in A or between

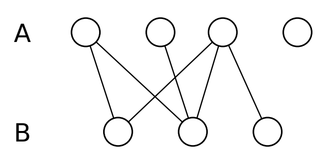
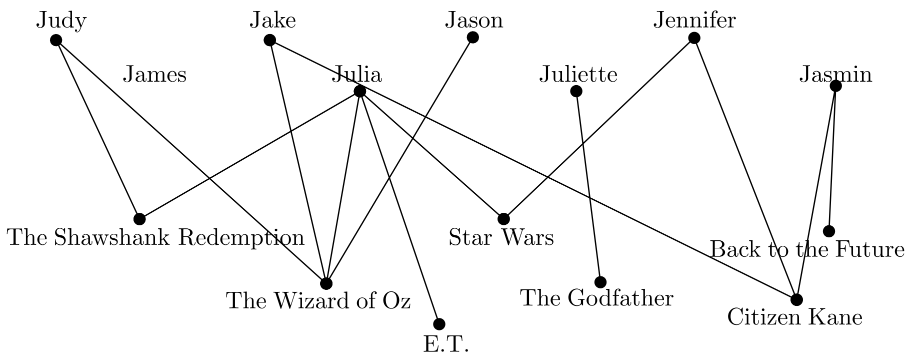


Figure 9: A bipartite graph with six edges on seven vertices.

pairs of vertices in B. One real-life example of a bipartite graph can be seen in the "Netflix Prize" problem. In 2009 Netflix offered a \$1 million dollar prize to the team that could best predict how much a viewer would enjoy a particular movie given their movie preferences. One could view this problem as studying a graph with two sets of vertices: viewers and movies. Edges are drawn between individuals who have viewed a particular movie in the past.



In such a graph it does not make sense to draw an edge between two people or between two movies. Much can be learned about this bipartite graph. We note that it should be clear that a bipartite graph need not be connected. This was clear in Figure 9 and is clear in this figure as well.

We might wonder about the largest number of edges in a bipartite graph. We recall that for a general graph on n vertices, we could have $n(n-1)/2$ edges. In the case of a bipartite graph, however, we should expect the number to be smaller, since there are no edges between vertices of each subset. If we want to add as many edges as possible, we should draw an edge from every vertex in A to every vertex in B. We will then have a total of $|A||B|$ edges. One way to see this is by noticing that the degree of every vertex in A is $|B|$ and the degree of every vertex in B is $|A|$. Using the formula $\sum \deg(v_i) = 2e = 2|E|$, we have $|A||B| + |B||A| = 2|E|$, which of course gives us $|E| = |A||B|$. If you don't find this convincing, experiment a bit using different sets A and B, and think about the maximal number of edges you can draw in such a graph.

5.2 Graph Isomorphism

Most properties of a graph do not depend on the particular names of the vertices. For example, although graphs A and B is Figure 10 are technically different (as their vertex sets are distinct), in some very important sense they are the "same"

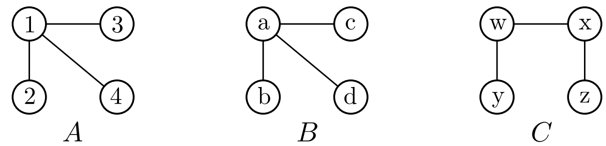


Figure 10: Two isomorphic graphs A and B and a non-isomorphic graph C; each have four vertices and three edges.

graph. For example, both graphs are connected, have four vertices and three edges. However, notice that graph C also has four vertices and three edges, and yet as a graph it seems different from the first two. Isomorphism is the idea that captures the kind of sameness that we recognize between A and B, and which distinguishes both of them from C.

Definition 19. *Two graphs G_1 and G_2 are* **isomorphic** *if there exists a matching between their vertices so that two vertices are connected by an edge in G_1 if and only if corresponding vertices are connected by an edge in G_2.*

In Figure 10, we can match the vertices of graph A with those of graph B in such a way: 1 is matched with a, 2 with b, 3 with c, and 4 with d. An edge connects 1 and 3 in the first graph, and so an edge connects a and c in the second graph. Likewise, no edge connects 3 and 4 in the first graph, and so no edge connects c and d in the second graph. Regarding the two graphs in Figure 10, we can write $A \cong B$ to denote this isomorphism. Although we matched vertices of A with those of B in one particular way, there could be several ways to do. For example, we could match 1 with a, 2 with c, 3 with d, and 4 with b; there are several other ways to do this. We often use the symbol $\cong$ to denote isomorphism between two graphs, and so would write $A \cong B$ to indicate that A and B are isomorphic.

Although graphs A and B are isomorphic, i.e., we can match their vertices in a particular way, graph C is not isomorphic to either of A or B. As hard as we try, we will fail to find a matching between vertices of A, for example, and those of C that maintain edge-connections between corresponding vertices. We can write $A \ncong C$ to indicate that A and C are not isomorphic.

Graph theorists are primarily interested in properties of graphs that do not change when vertices are relabeled; sometimes they will discuss "properties that are invariant under isomorphisms" which conveys this idea. We note that if $G_1 \cong G_2$, then many properties of G_1 and G_2 must be the same. For example, the number of vertices and edges in the two graphs must be identical. G_1 is *connected* if and only G_2 is connected. G_1 is k-regular if and only if G_2 is k-regular. G_1 is bipartite if and only if G_2 is bipartite. The numbers of vertices with degree 0, 1, 2, etc. must be identical. None of the properties listed here change when vertices are relabeled. Conversely, if two graphs G_1 and G_2 differ with respect to any of these properties, then we can know that G_1 and G_2 are not isomorphic.

Many properties of individual vertices also do not change "under isomorphisms", or relabeling of the vertices. For example, if vertex u in graph G_1 can be matched with vertex v in G_2, then we must have $\deg(u) = \deg(v)$. If $\deg(u) \neq \deg(v)$, then we cannot match up the two vertices.

Determining whether two graphs are isomorphic is not always an easy task. For graphs with only several vertices and edges, we can often look at the graph visually to help us make this determination. In the following pages we provide several examples in which we consider whether two graphs are isomorphic or not. Our focus here is more on visual presentations of graphs, but we could also consider presentations of graphs in terms of sets.

Example 1

A relabeling of vertices of a graph is isomorphic to the graph itself. Consider the three isomorphic graphs illustrated in Figure 11. The first two graphs illustrate

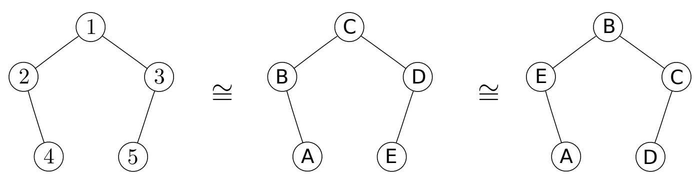


Figure 11: Three isomorphic graphs.

a change of using letters to using numbers to label the graphs. The second pair of graphs are also isomorphic as only the labels were changed. We can match vertices in the second graph with those in the third graph to satisfy the isomorphism requirements. Another way to think about graph isomorphism is by removing all vertex labels from two graphs. It is clear for these examples that all three graphs are then identical.

Example 2

Like relabeling, moving around vertices also does not change important graph properties. The two graphs illustrated in Figure 12 are isomorphic since edges connected in one are also connected in the other. In fact, not only are the graphs isomorphic to one another, but they are in fact identical. Notice that each vertex in one graph is matched to itself in the other graph.

Example 3

The figure below illustrates another pair of isomorphic graphs. Although the graphs have a slightly different shapes from one another, we can still find a

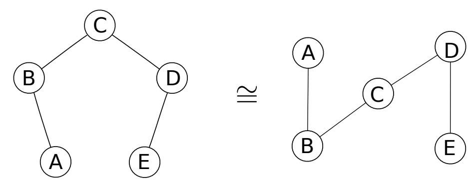


Figure 12: Two isomorphic graphs.

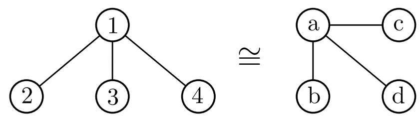


Figure 13: Two isomorphic graphs.

1-1 matching between the vertices so that if pairs of vertices are connected by an edge in one graph, then corresponding vertices will be connected in the other. The same matching given above (a1, b2, c3, d4) will still work here, even though we have moved the vertices around. It is worth noting that several other matchings would also work. For example a1, b3, c4, d2 would also be a good matching. In fact, in this example, as long as a is matched with 1, then b, c, and d can be matched in any order with 2, 3, and 4.

Example 4

Any two complete graphs K_n and K_m are isomorphic if any only if $n = m$. If $n \neq m$, then it is clear that we cannot have a 1-1 matching between the vertices of the two graphs, because there will be more vertices in one graph than in the other. If $n = m$ then any matching will work, since all pairs of distinct vertices are connected by an edge in both graphs. Notice that in the graphs below, any matching of the vertices will ensure the isomorphism definition is satisfied.

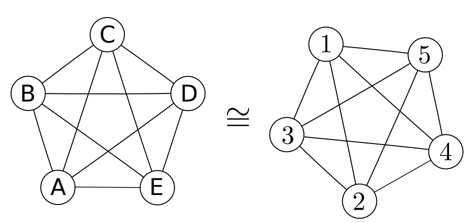


Figure 14: Two complete graphs on five vertices; they are isomorphic.

Example 5

Just because two graphs have the same number of vertices and edges does not mean that they are isomorphic. In fact, even if the degrees of all vertices are identical, still the two graphs can be non-isomorphic. Although $G_1 \cong G_2$ (we

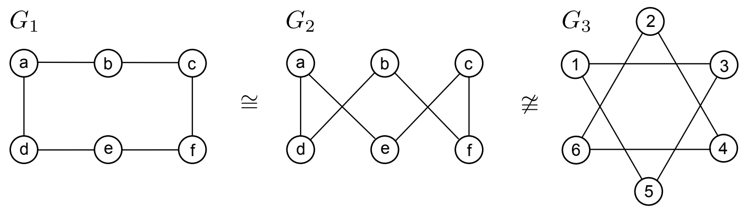


Figure 15: Three 2-regular graphs on six vertices; the first two are isomorphic; the third one is not.

can imagine deforming either into the other), G_2 and G_3 are not isomorphic. No matter how we relabel the vertices of G_2, it will remain a connected graph. Likewise, no matter how we relabel the vertices of a G_3, it will remain unconnected. Two graphs that are isomorphic must both be connected or both disconnected.

Example 6

Below are two complete graphs, or cliques, as every vertex in each graph is connected to every other vertex in that graph. As a special case of Example 4,

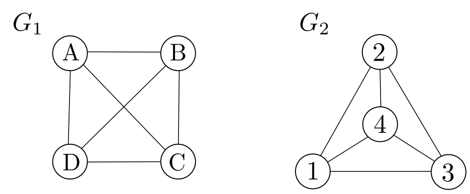


Figure 16: Two complete graphs on four vertices; they are isomorphic.

we already know that these two graphs are isomorphic since they have the same number of vertices. The two drawing here, however, highlight a particularly interesting feature of certain graphs. We have already seen that we can generally draw graphs in many different ways without changing their overall structure. So long as we don't disconnect any vertices that are connected to each other, and so long as we don't attach any vertices that were previously disconnected, we are free to move the vertices and edges as we please. In this example, in the first way we drew the graph, two of its edges (AC and BD) crossed one another. However, in the second drawing, of an isomorphic graph, we were able to draw the graph in such a way that no two edges cross. In other words, there are some graphs that can be drawn without edges crossing. Can *all* graphs be drawn in such a way that no two edges cross? This question leads us to consider the next big topic in graph theory, the study of planarity.

5.3 Planar Graphs and Euler's Formula

Among the most ubiquitous graphs that arise in applications are those that can be drawn in the plane without edges crossing. For example, let's revisit the example considered in Section 5.1 of the New York City subway system. We considered a graph in which vertices represent subway stops and edges represent direct train routes from one subway stop to the next. We might wonder whether such a graph can be drawn without any edges crossing. Why should we care about this? Since each edge represents a subway line, the crossing of edges represents two subway lines that cross paths. For that to happen, either train tracks will need to cross, an engineering feat involving careful consideration of potentially-conflicting schedules and the engineering of special tracks, or else subway lines must be built at different depths below ground. Both of these options are costly and challenging. It turns out that the graph of the NYC subway system cannot be drawn in such a way, and indeed many subway lines run at different depths below ground in various parts of the city.

In this section we consider which graphs can be drawn on paper without edges crossing and which graphs cannot.

Definition 20. *A graph G is* **planar** *if it can be drawn in the plane in such a way that no pair of edges cross.*

Attention should be paid to this definition, and in particular to the word 'can'. Whether or not a graph is planar does *not* depend on how it is actually drawn. Instead, planarity depends only on whether it 'can' be drawn in such a way. By defining this property in this more abstract way, we can ensure that planarity is preserved under isomorphisms. If planarity depended on how a particular graph was drawn, then we could have two isomorphic graphs, such that one is planar and the other is not. Furthermore, graphs that are only described abstractly through a vertex set V and an edge set E, and without being drawn, could not be described as planar or not, since there could be multiple ways of drawing it.

Determination of Planarity

Sometimes it is easy to see that a particular graph is planar, especially when it is drawn in such a way. In Examples 1, 2, and 3 of Section 5.2, we can readily see that all graphs are planar, as no edges cross. However, we have noted in the discussion of Example 6 that it is sometimes difficult to determine that a particular graph is planar just from looking at it. For example, G_1 in Example 6 of Section 5.2 might give the mistaken impression that K_4 is a non-planar graph, even though G_2 there makes clear that it is indeed planar; the two graphs are isomorphic. These observations motivate the question of whether there exists a way of looking at a graph and determining whether it is planar or not.

Euler's Formula for Planar Graphs

The most important formula for studying planar graphs is undoubtedly Euler's formula, first proved by Leonhard Euler, an 18$^{\text{th}}$ century Swiss mathematician, widely considered among the greatest mathematicians that ever lived. Until now we have discussed vertices and edges of a graph, and the way in which these pieces might be connected to one another. In a sense, vertices are 0-dimensional pieces of a graph, and edges are 1-dimensional pieces. In planar graphs, we can also discuss 2-dimensional pieces, which we call faces. Faces of a planar graph are regions bounded by a set of edges and which contain no other vertex or edge.

Example 1

Several examples will help illustrate faces of planar graphs. The figure below

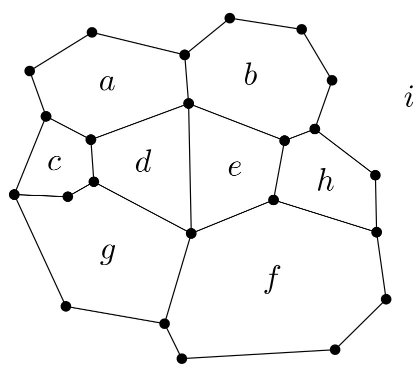


Figure 17: A planar graph with faces labeled using lower-case letters.

illustrates a planar graph with several bounded regions labeled a through h. These regions are called faces, and each is bounded by a set of vertices and edges. For reasons that will become clear later, we also count the region "outside" of the graph as a face; we sometimes call this the "outside" face.

Euler discovered a beautiful result about planar graphs that relates the number of vertices, edges, and faces. In what follows, we use $v = |V|$ to denote the number of vertices in a graph, $e = |E|$ to denote the number of edges in a graph, and f to denote its number of faces. Using these symbols, Euler's showed that for any connected planar graph, the following relationship holds:

$$v - e + f = 2. \tag{47}$$

In the graph above in Figure 17, $v = 23$, $e = 30$, and $f = 9$, if we remember to count the outside face. Indeed, we have $23 - 30 + 9 = 2$. This relationship holds for all connected planar graphs.

Example 2

An infinite set of planar graphs are those associated with polygons. The figure below illustrates several graphs associated with regular polyhedra. Of course,

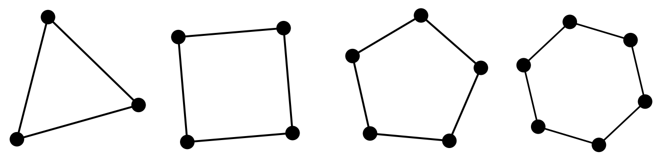


Figure 18: Regular polygonal graphs with 3, 4, 5, and 6 edges.

each graph contains the same number of edges as vertices, so $v - e + f = 2$ becomes merely $f = 2$, which is indeed the case. One face is "inside" the polygon, and the other is outside.

Example 3

A special type of graph that satisfies Euler's formula is a tree. A tree is a graph such that there is exactly one way to "travel" between any vertex to any other vertex. These graphs have no circular loops, and hence do not bound any faces. As there is only the one outside face in this graph, Euler's formula gives us

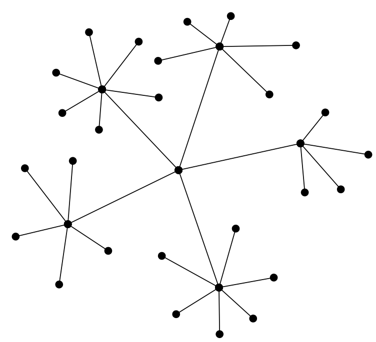


Figure 19: A tree graph – there are no faces except for the outside one.

$v - e + 1 = 2$, which simplifies to $v - e = 1$ or $e = v - 1$. Every tree satisfies this relationship and so always has one fewer edges than it has vertices.

Degree of a Face

In the same way that we were able to characterize a vertex by counting the number of edges adjacent to it, we can also characterize a face by its number of edges (or equivalently vertices) on its boundary. For example, consider Figure 20, which shows a graph we considered earlier. Here each face is labeled by its

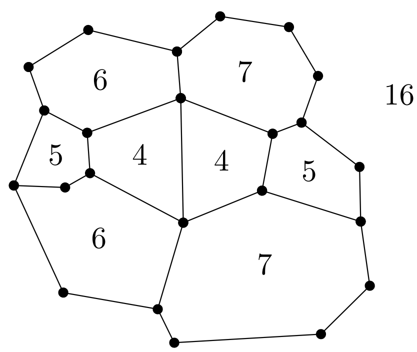


Figure 20: A planar graph with each face labeled by its degree.

number of edges. We use the word *degree* to refer to the number of edges of a face.

Definition 21. *The* **degree** *of a face f is the number of edges along its boundary. Alternatively, it is the number of vertices along its boundary. Alternatively, it is the number of other faces with which it shares an edge. The degree of a vertex f is oftentimes written* $\deg(f)$.

Every edge in a planar graph is shared by exactly two faces. This observation leads to the following theorem.

Theorem 8. *For all planar graphs, the sum of degrees over all faces is equal to twice the number of edges. In symbols,* $\sum_i \deg(f_i) = 2|E|$*, where f_i are the faces of the graph.*

This result might be seen as an analogue of a result we saw earlier involving the sum of degrees of all vertices (Theorem 7). These theorems help us understand the relationship between the number of edges in a graph and the vertices and faces of a (planar) graph.

Our definition of a graph (as a set V and a set E consisting of two-element subsets of V) requires that there be at most one edge connecting any two vertices. To understand this, consider two vertices a and b. We have learned before (in Section 2) that sets do not have duplicate elements. Therefore, the set of edges E can only contain the element $\{a, b\}$ one time. Such graphs are called *simple* and they have been the exclusive focus of our consideration in this section. Because our graphs are all simple, the smallest possible degree of a face is 3, since a face with degree two would require that two edges connect a pair of vertices.

We can use this result, along with Theorem 9, to obtain an important result about planar graphs. In particular, notice that since the degree of every face must be at least 3, we have $\sum_i 3 \leq \sum_i \deg(f_i)$. The left-hand side, however, evaluates to $3|F|$, where $|F|$ is the number of faces in the planar graph, because we are adding 3 for every face. Theorem 9 tells us that the right-hand side is equal to $2|E|$, where $|E|$ is the number of edges in the graph. Therefore, we can conclude:

Theorem 9. *For all planar graphs,* $3|F| \leq 2|E|$*, where* $|F|$ *is the number of faces and* $|E|$ *is the number of edges.*

This, of course, is equivalent to stating that $|E| \geq \frac{3}{2}|F|$; the number of faces of a planar graph ensures that we have at least a certain number of edges.

Non-planarity of K_5

We can use Euler's formula to prove that non-planarity of the complete graph (or clique) on 5 vertices, K_5, illustrated below. This graph has $v = 5$ vertices

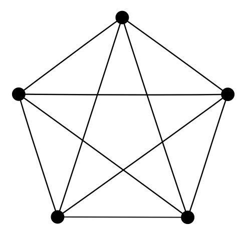


Figure 21: The complete graph on five vertices, K_5.

and $e = 10$ edges, so Euler's formula would indicate that it should have $f = 7$ faces. We have just seen that for any planar graph we have $e \geq \frac{3}{2}f$, and so in this particular case we must have at least $\frac{3}{2}7 = 10.5$ edges. However, K_5 only has 10 edges, which is of course less than 10.5, showing that K_5 cannot be a planar graph.

Faces in Non-planar Graphs

Non-planar graphs do not technically have faces – there does not seem to be any good way to discuss faces in cases when edges cross one another.

5.4 Polyhedral Graphs and the Platonic Solids

Regular Polygons

In this section we will see how Euler's formula – unquestionably the most important theorem about planar graphs – can help us understand polyhedra and a special family of polyhedra called the Platonic solids. You might recall that *polygons* are two dimensional shapes such as triangles, rectangles, pentagons, and hexagons. Below are illustrated polygons with 3, 4, 5, and 6 edges. Each

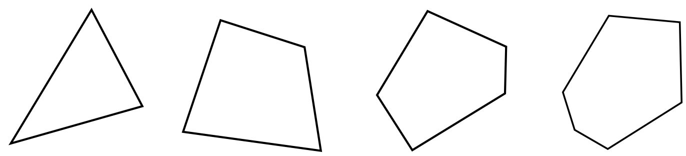


Figure 22: Polygons with 3, 4, 5, and 6 edges.

of these shapes is constructed using three or more straight line segments connected together at their endpoints. The lengths of these line segments are not necessarily the same for each side of the polygon, nor are the internal angles at which pairs of edges meet. Such shapes are called *irregular polygons*, or just *polygons*.

If all edges have the same length, and all pairs of edges meet at identical angles, then such shapes are called *regular polygons*. Regular polygons with 3, 4, 5, and 6 edges are illustrated below. Although most regular polygons

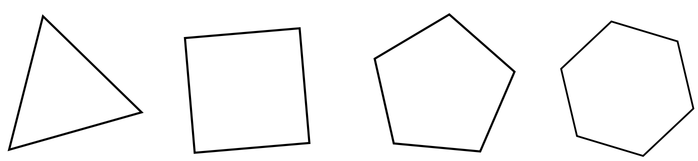


Figure 23: Regular polygons with 3, 4, 5, and 6 edges.

we encounter do not have many sides, we can construct a regular polygon for any number of sides greater than 2. The figure below illustrates several regular polygons with large numbers of edges. Although it might be difficult to construct

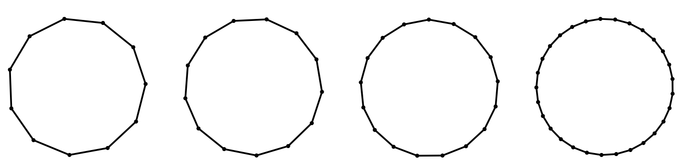


Figure 24: Regular polygons with 11, 13, 17, and 29 edges; small circles placed at corners to make edges more visible.

or count its number of edges, we can construct regular polygons with a hundred, a thousand, or even a million edges.

Polyhedra

Polyhedra are three-dimensional analogues of polygons. Instead of being constructed of line segments, polyhedra are constructed from polygons connected together along edges. The polygons do not necessarily need to be regular. Below are illustrated several polyhedra composed of different numbers of polygonal faces. The reader might notice that the first and third example stand out as

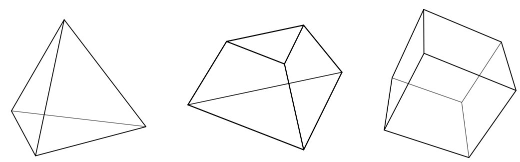


Figure 25: Several familiar polyhedra with 4, 5, and 6 faces.

being particularly symmetric. Indeed, in these two cases, each face of the polyhedron is an identical regular polygon. Moreover, the same number of faces meet at every corner. Such polyhedra are called *regular*, and can be considered three-dimensional analogues of two-dimensional regular polygons.

Despite some similarities between regular polygons and regular polyhedra, there turns out to be a fundamental difference in how many of them we can find. In particular, although for every $n \geq 3$ there exists a regular polygon with n sides, there are only five values of n for which there exists a regular polyhedron with n faces. These are known as the Platonic solids, and Euler's theorem will help us enumerate their possibilities.

Polyhedral Graphs

In order to make Euler's theorem useful in studying polyhedra, we need to understand the relationship between polyhedra and planar graphs. We begin by noting that every polyhedron uniquely determines a graph up to isomorphism. To see this, we place a vertex at every corner of a polyhedron. If we consider the cube, for example, we can construct a graph that has 8 vertices, one corresponding to each corner. Next, we connect pairs of vertices if both lie along the same edge in the polyhedron. Graphs constructed in this manner are called **polyhedral graphs**. If we wrote out this graph in terms of its vertex set V and edge set E, we would have:

$$\begin{aligned}
V &= \{a,b,c,d,e,f,g,h\} \\
E &= \{\{a,b\},\{b,c\},\{c,d\},\{d,a\}, \\
&\quad \{e,f\},\{f,g\},\{g,h\},\{h,e\}, \\
&\quad \{a,e\},\{b,f\},\{c,g\},\{d,h\}\}.
\end{aligned}$$

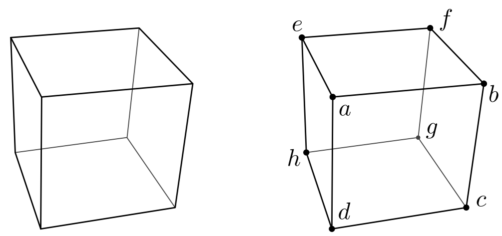


Figure 26: A cube and its associated graph.

Not only can we construct a graph using the corners and edges of a polyhedron, but all such graphs turn out to be planar graphs – this is a subtle point worth considering before continuing. One way to think about why this is true is by considering what happens when we choose one face of the polyhedron and "stretch it out"; we call this face the "outside" face for reasons that will become clear in a moment. For example, in the example above, consider what happens when we take the face *abcd* and stretch it out. In this example, we can move

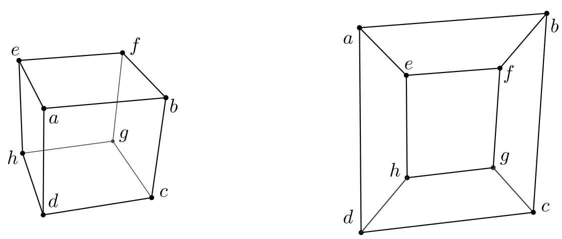


Figure 27: Moving vertices of the polyhedral graph to illustrate that it is planar; we "stretched out" the vertices a, b, c, and d, and "moved" all remaining vertices inside it.

out the vertices a, b, c, and d, and move in the remaining vertices. Since this graph is now drawn without any edges crossing one another, it is clear that the graph associated with the cube is indeed planar. The face *abcd* is now drawn on the outside of the graph, thus justifying its name. We could have chosen any one of the six faces to be the outside face, though in each case the planar graph we drew would have looked the same.

While in the case above, which face we choose to be the "outside" face makes no difference to the picture of the planar graph (aside from vertex labelings), this is not generally the case. Consider for example the triangular prism, illustrated below with an associate polyhedral graph. As a polyhedral graph, we can draw this is in several ways. If we choose one of the triangular faces, such as *abc* as the outside face, we will obtain a drawing such as the one on the left of Figure 29; if we choose a quadrilateral face, such as *abed*, then we will obtain a drawing such as that on the right. These graphs are identical as graphs.

Although the graph associated with every polyhedron is a planar graph,

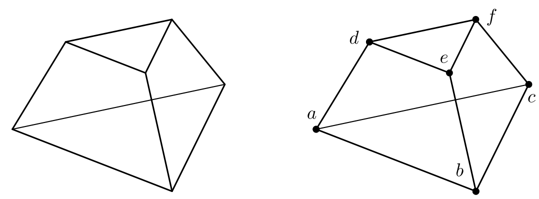


Figure 28: Triangular prism and an associated polyhedral graph.

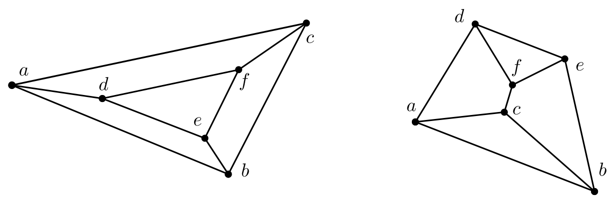


Figure 29: Two planar drawings of the polyhedral graph associated with the triangular prism.

there exist planar graphs that are not graphs of polyhedra. Although we will not consider examples of such here, it is a point worth thinking about. Can you think of a planar graph that is not the graph of any polyhedron?

The Platonic Solids

Euler's formula allows us to use what we know about planar graphs to prove that there exist only five regular polyhedra. For our purposes, we consider the following definition:

Definition 22. *A* **regular polyhedron** *is one in which all faces are identical regular polygons, and such that the same number of faces meet at every corner.*

In terms of planar graphs, this means that every face in the planar graph (including the outside one) has the same degree (number of edges on its boundary), and every vertex has the same degree. We are now able to prove the following theorem.

Theorem 10. *There are no more than 5 regular polyhedra.*

Proof. In proving this theorem we will use n to refer to the number of edges of each face of a particular regular polyhedron, and d to refer to the degree of each vertex. We will show that there are only five different ways to assign values to n and d that satisfy Euler's formula for planar graphs.

Let us begin by restating Euler's formula for planar graphs. In particular:

$$v - e + f = 2. \tag{48}$$

In this equation, v, e, and f indicate the number of vertices, edges, and faces of the graph. Previously we saw that if we add up the degrees of all vertices in a graph (not necessarily planar) we obtain a number that is twice the number of edges. In equation form, we have

$$\sum_{i}^{v} \deg(v_i) = 2e,$$

where v_i is the ith vertex. In our case, since we are only considering graphs in which each vertex has the same degree d, we can rewrite this as $vd = 2e$, or

$$v = 2e/d. \tag{49}$$

We have also seen that if we add up the degrees of all faces in a planar graph we obtain a number that is twice the number of edges. In equation form, we have

$$\sum_{i}^{f} \deg(f_i) = 2e,$$

where f_i is the ith face. In our case, since we are only considering graphs in which each face has the same degree n, we can rewrite this as $fn = 2e$, or

$$f = 2e/n. \tag{50}$$

Equations 48, 49, and 50 provide us with sufficient information to prove that there are at most five regular polyhedra. First, we can substitute Equations 49 and 50 into Equation 48 to obtain:

$$\frac{2e}{d} - e + \frac{2e}{n} = 2. \tag{51}$$

We can rearrange the terms and divide all of them by $2e$ to obtain

$$\frac{1}{n} + \frac{1}{d} = \frac{1}{2} + \frac{1}{e}. \tag{52}$$

Since the number of edges in a polyhedral graph is always positive, the term $\frac{1}{e}$ must be positive. Therefore, if we remove the term $\frac{1}{e}$ from the right-hand side, we make it smaller than the left-hand side.

$$\frac{1}{n} + \frac{1}{d} > \frac{1}{2}. \tag{53}$$

This inequality, which must be true for every regular polyhedral graph, tells us about the possible values of n and d. First, notice that if n and d are both very large, then the left-hand side will be very small. For example, notice that if $n = 4$ and $d = 4$, then we obtain the false inequality:

$$\frac{1}{4} + \frac{1}{4} > \frac{1}{2}. \tag{54}$$

Since this is not true, at least one of n and d must be smaller than 4. However, notice also that neither of n or d can be smaller than 3, since faces cannot have fewer than 3 edges, and vertices, in polyhedral graphs, cannot have degree smaller than 3 (think about this). Therefore, for any regular polyhedron, at least one of n or d must be exactly 3.

Let us consider each of the two cases individually. We begin with $n = 3$, or polyhedral graphs in which all faces have three edges, i.e., all faces are triangles. Substituting $n = 3$ into Equation 53, we find out that $\frac{1}{d} > \frac{1}{6}$, or that $d < 6$. This leaves us with three options, either $d = 3$, 4, or 5. This gives us three possible regular polyhedra entirely of triangles.

Alternatively, we consider what happens when we require that $d = 3$. Similar calculations show that this forces n to be either 3, 4, or 5. This too gives us three possible regular polyhedra, all of whose vertices have degree 3. Notice that there is one overlap between the two sets of polyhedra – the one all of whose faces are triangles and all of whose vertices have degree 3. Therefore, there are only five unique pairs of n and d that can describe regular polyhedra.

Each of these five choices of n and d results in a different regular polyhedron, illustrated below. $\square$

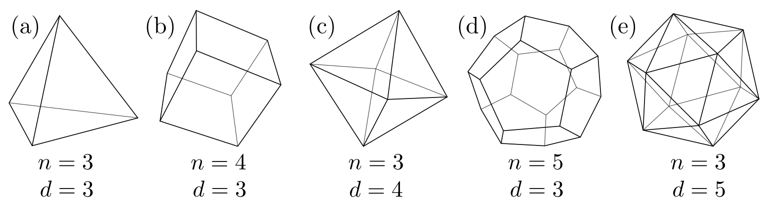


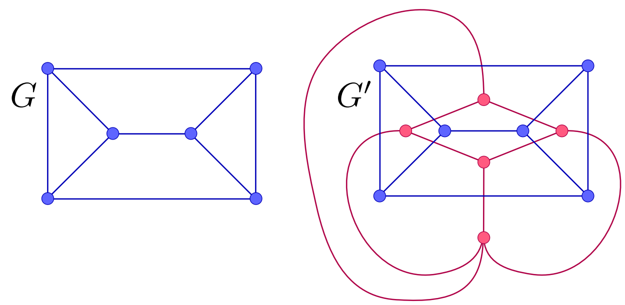
Figure 30: The five regular polyhedra, also known as the Platonic solids. Below are listed the numbers of vertices v, edges e, and faces f of each regular polyhedron, as well as the number of edges per face n and degree d of each vertex.

Name of Polyhedron	v	e	f	d	n
(a) tetrahedron	4	6	4	3	3
(b) cube	8	12	6	3	4
(c) octahedron	6	12	8	4	3
(d) dodecahedron	20	30	12	3	5
(e) icosahedron	12	30	20	5	3

Dual Graphs and Dual Polyhedra

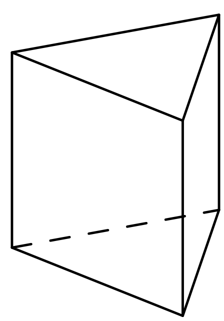
A beautiful topic that arises in many areas of pure and applied mathematics is that of *dual* objects or problems. That is, we might be given a particular problem or object, and to study it, it is best to study another problem that is intimately related to the first problem. Here we briefly consider dual graphs and dual polyhedra.

Think for a moment about how we might begin with a planar graph and use it to generate a new one. In particular, think about what happens when we take a planar graph G and use it to make a new graph G' (pronounced G prime) in the following manner. For every face in G we make a vertex in the new graph G'. We then connect pairs of vertices in G' if corresponding faces in G share an edge.



Above to the left is illustrated a graph G with five faces, including the outside face; this graph is colored blue. We make a new graph that we call G' based on G, as described above. Notice that for every face with degree 3 in G there is a vertex with degree 3 in G', and for every face with degree 4 in G there is a vertex with degree 4 in G'. Notice also that we create one edge in G' for every edge in G.

We could have also performed this construction on polyhedra instead of on planar representations of them. For example, consider the triangular prism



illustrated above. Indeed, this graph is isomorphic to G above. We can construct a new polyhedra by placing one vertex at the center of each face, and then connected vertices whose corresponding faces share an edge. This is called the dual polyhedra. What would it look like?

[Notes here are incomplete; additional information on this subject can be found on https://en.wikipedia.org/wiki/Dual_graph]

5.5 Map Colorings

In Section 5.4 we considered an application of graph theory for studying polyhedra. In particular, we used Euler's formula to prove that there can be no more than five regular polyhedra, which are known as the Platonic Solids. Many classical philosophers believed in a mystical correspondence between these polyhedra and air, earth, fire, and water – which they understood to be the four basic elements of the world; the fifth polyhedron corresponds to the universe itself, or to the 'aether'. This belief is no longer widespread, but it might be of some historical or cultural interest to some readers.

We now consider an application of graph theory, and of Euler's formula, in studying the problem of how maps can be colored. Map-makers often color adjacent geo-political regions differently, so that map-readers can quickly distinguish distinct regions. In the illustration below on the left, we color Pennsylvania orange, West Virginia yellow, New York purple, and so forth. If we had a box of 64 Crayola crayons, of course we would have enough colors so that every state could have a distinct color. But if we only have five or six colors, we certainly can't color every state a different color. But maybe we can still color every *adjacent* state a different color. Is that possible? Or, to make this question a bit more precise – how many colors would we need to make sure that adjacent states never share a color? This is a classical problem in graph theory, and in this section we'll use graph theory, and in particular planar graphs and Euler's formula, to study it.

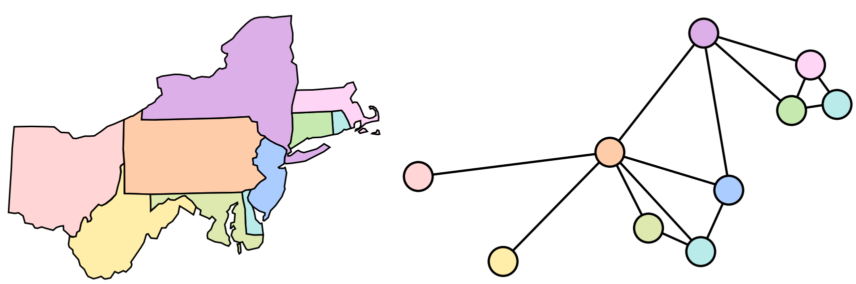


Figure 31: Map of several northeastern states, and a representation of this map as a planar graph.

To see how graphs can be relevant to studying maps, we construct a new graph so that each state is represented by a vertex, and so that two vertices are connected by an edge if and only if the two states share a boundary. The illustration above on the right shows such a graph. Although it is not entirely obvious, such a graph is always planar, as crossing edges would indicate crossing borders between adjacent states, which cannot occur.

Since maps can be represented as planar graphs, if we can prove that some number of colors is always sufficient to color the vertices of a planar graph, then we can also know that that number of colors is sufficient to color a map. In what follows, we will prove that 6 colors is always sufficient to color a planar graph. In fact, 4 colors is also sufficient to color any planar graph, but proving that statement is quite involved, and would take the remainder of the semester to investigate. The two papers that proved this theorem in 1976 required well over a hundred pages, and are well beyond the treatment here. However, we can still prove that 6 colors are sufficient. Proving that 5 colors is sufficient is more difficult than proving that 6 are sufficient, but nowhere nearly as difficult as proving that 4 are sufficient.

Vertices with Small Degree

In order to prove that every planar graph can be colored with 6 colors, we first need to prove the following theorem:

Theorem 11. *Every planar graph contains at least one vertex with degree at most 5.*

Proof. We have previously seen that for an arbitrary graph with v vertices and e edges, we can calculate e by considering the degrees of all vertices v_i. In particular,

$$\sum_{i=1}^{v} \deg(v_i) = 2e. \tag{55}$$

We can use this relationship to write the average degree of a vertex in terms of e and v. If we take the sum of the degrees and divide that sum by the total number of vertices, then we can write this average, which we call $\overline{\deg}$, as:

$$\overline{\deg} = 2e/v. \tag{56}$$

We have also considered a similar relationship between the number of edges in a graph and the degrees of its faces. If we take into account the outside face of a planar graph, then every edge in a planar graph appears in exactly two faces. If a graph has f faces, and if we use $\deg(f_i)$ to refer to the number of edges around face f_i, then we can write:

$$\sum_{i=1}^{f} \deg(f_i) = 2e. \tag{57}$$

Since every face must have at least 3 faces (i.e., $\deg(f_i) \geq 3$ for all faces), then we can rewrite Equation 57 as an inequality

$$3f \leq 2e, \tag{58}$$

or equivalently as

$$f \leq \frac{2}{3}e. \tag{59}$$

We should recall here Euler's formula for planar graphs which can be written as:

$$f = 2 + e - v. \tag{60}$$

Combining the previous two equations, we have:

$$2 + e - v \leq \frac{2}{3}e. \tag{61}$$

Basic high-school algebra allows us to rearrange this and conclude that:

$$2e/v \leq 6 - \frac{12}{v}. \tag{62}$$

We have already seen above that $2e/v$ is equal to the average degree of a vertex. In other words, we have:

$$\overline{\deg} \leq 6 - \frac{12}{v}. \tag{63}$$

Since v is always a positive number, the quantity $12/v$ is also always positive, and so the right-hand side of Equation 63 is a number strictly smaller than 6. This shows that at least one vertex must have degree smaller than 6, since if the degree of every vertex was 6 or greater, then this average would be 6 or larger. We thus conclude that every planar graph has at least one vertex with degree at most 5. □

Every Planar Graph is 6-colorable

Knowing that every planar graph has at least one vertex with degree at most 5 allows us to prove that:

Theorem 12. *The vertices of every planar graph can be colored using 6 colors in such a way that no pair of vertices connected by an edge share the same color.*

Proof. We begin by noticing that every graph on 6 or fewer vertices can certainly be colored with 6 colors, since we can color each vertex with a different color. Our challenge is then to consider what happens when we have a graph with 7 or more vertices. Can all graphs with 7 or more vertices be colored with only 6 colors? We use a technique called mathematical induction to show that the answer to this question is yes. In particular, we show that if every planar graph with k vertices can be colored using 6 colors, then so too can every planar graph with $k + 1$ vertices. For example, we will show that if all graphs with $k = 6$ vertices can be colored with 6 colors, then so can all graphs with $k + 1 = 7$ vertices. By proving this, we in effect show that not only can every graph with 6 vertices be colored using 6 colors, but so too can every graph with 7 vertices, and every graph with 8 vertices, and every graph with 9 vertices, and so forth.

How do we prove that if every planar graph with k vertices can be colored with 6 colors, then so too can every graph with $k + 1$ colors? Let's consider a hypothetical graph G on $k + 1$ vertices; part of such a graph is illustrated in Figure 32. We want to show that G can be colored using at most 6 colors. From Theorem 11 we know that G must have at least one vertex with degree at most 5. Let us find one of those vertices and call it s. For a moment, let's consider what happens when we remove vertex s. We construct a new graph

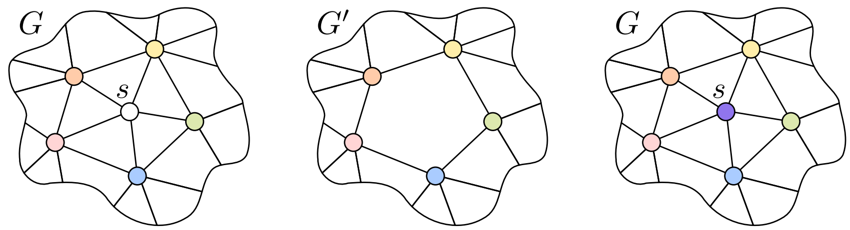


Figure 32: Parts of a graph G with $v = k+1$ vertices, and of a modified version, which we call G', obtained by removing the vertex s. After putting back s, it is clear that we can color it using a color not used by any of its neighbors.

which is identical to G except that s is now removed; we call this new graph G', to indicate that it's a modified version of G. Since G had $k+1$ vertices and we removed one vertex to create G', then G' must have k vertices. Since we already know that all graphs with k vertices can be colored with 6 colors, we in effect know that G' can be colored using only 6 colors.

What happens when we try to put vertex s back into the graph G'? We know that s has at most 5 neighbors, meaning that at most 5 other colors are being used to color vertices adjacent to s. This means that we can find a sixth color not used by any neighbor of s, and which we can use to color s. Upon restoring s to our graph and coloring it that final color, we obtain our original graph G which is now colored using at most 6 colors.

This shows that if all planar graphs with k vertices can be colored using 6 colors, then so can all planar graphs with $k+1$ vertices. Since we know that all graphs on 6 vertices can be colored with at most 6 colors, we then also know that all graphs with 7 vertices, and 8 vertices, and 9 vertices, etc, can also be colored in such a way. This completes our proof by induction of the theorem. $\square$

It is also possible to prove in a reasonable short space that every planar graph can be colored with only 5 colors, though we do not consider that proof here. As we have noted earlier, it is actually true that every map can be colored using only 4 colors, but the proof of that statement is very complicated and well beyond the tools we have developed so far.

5.6 Euler Paths and Cycles

One of the oldest and most beautiful questions in graph theory originates from a simple challenge that can be played by children. The town of Königsberg (now

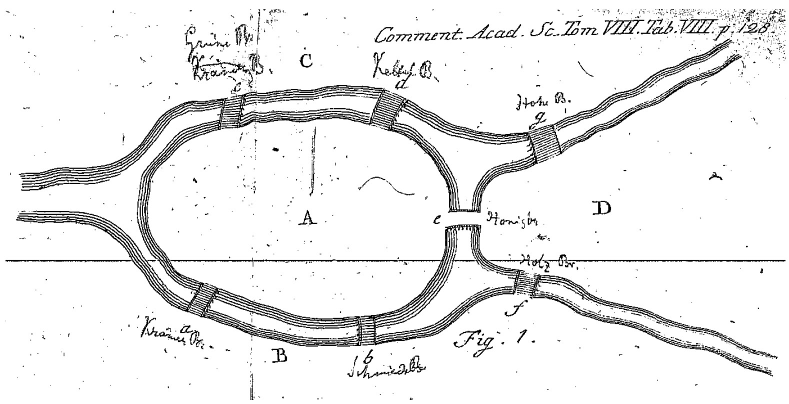


Figure 33: An illustration from Euler's 1741 paper on the subject.

Kaliningrad, Russia) is situated near the Pregel River. Residents wondered whether they could they begin a walk in one part of the city and cross each bridge exactly once. Many tried and many failed to find such a path, though understanding why such a path cannot exist eluded them.

Graph theory, which studies points and connections between them, is the perfect setting in which to study this question. Land masses can be represented as vertices of a graph, and bridges can be represented as edges between them. Generalizing the question of the Königsberg residents, we might ask whether for a given graph we can "travel" along each of its edges exactly once. Euler's work elegantly explained why in some graphs such trips are possible and why in some they are not.

Definition 23. *A* **path** *in a graph is a sequence of adjacent edges, such that consecutive edges meet at shared vertices. A path that begins and ends on the same vertex is called a* **cycle***.*

Note that every cycle is also a path, but that most paths are not cycles. Figure 34 illustrates K_5, the complete graph on 5 vertices, with four different paths highlighted; Figure 35 also illustrates K_5, though now all highlighted paths are also cycles.

In some graphs, it is possible to construct a path or cycle that includes every edges in the graph. This special kind of path or cycle motivate the following definition:

Definition 24. *An* **Euler path** *in a graph G is a path that includes every edge in G; an* **Euler cycle** *is a cycle that includes every edge.*

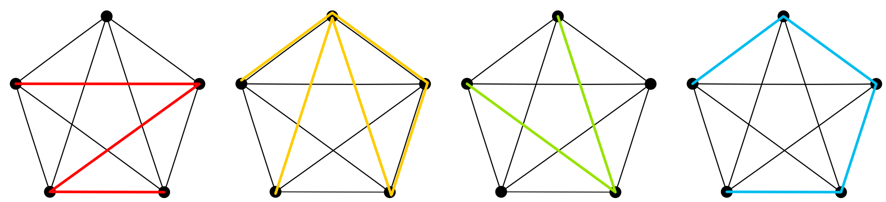


Figure 34: K_5 with paths of different lengths.

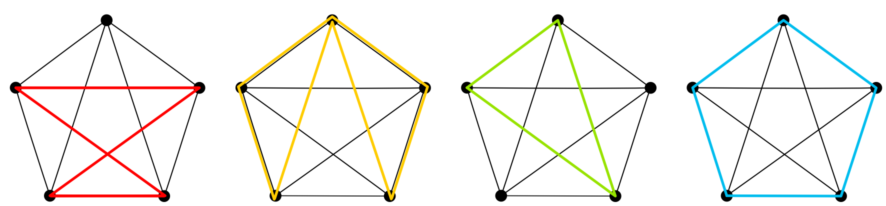


Figure 35: K_5 with cycles of different lengths.

Spend a moment to consider whether the graph K_5 contains an Euler path or cycle. That is, is it possible to travel along the edges and trace each edge exactly one time. It turns out that it is possible. One way to do this is to trace the (five) edges along the boundary, and then trace the star on the inside. In such a manner one travels along each of the ten edges exactly one time. One also ends at the same point at which one began, and so this Euler path is also an Euler cycle.

This example might lead the reader to mistakenly believe that every graph in fact has an Euler path or Euler cycle. It turns out, however, that this is far from true. In particular, Euler, the great 18th century Swiss mathematician and scientist, proved the following theorem.

Theorem 13. *A connected graph has an Euler cycle if and only if all vertices have even degree.*

This theorem, with its "if and only if" clause, makes two statements. One statement is that if every vertex of a connected graph has an even degree then it contains an Euler cycle. It also makes the statement that only such graphs can have an Euler cycle. In other words, if some vertices have odd degree, the the graph cannot have an Euler cycle. Notice that this statement is about Euler cycles and not Euler paths; we will later explain when a graph can have an Euler path that is not an Euler cycle.

Proof. How can show that every graph with an Euler cycle has no vertices with odd degree? One way to do this is to imagine starting from a graph with no edges, and "traveling" along the Euler cycle, laying down edges one at a time, until we have constructed our original graph. We consider what happens to the degree of each vertex as we travel around the graph adding edges. Notice that before doing any traveling, and so before we draw in any of the edges, the degree of each vertex is 0. Let us now consider the vertex from which we start and call it v_0. After leaving v_0 and laying down the first edge, we have increased the degree of v_0 by 1, i.e., $\deg(v_0) = 1$. So long as we don't return to v_0, its degree will stay 1. Now, notice what happens as we travel along our graph adding edges. Every time we pass through a vertex, we increase its degree by 2. The reason for this is that every time we pass through a vertex, we add one degree for the edge "entering" it and one degree for the edge "exiting" it. The

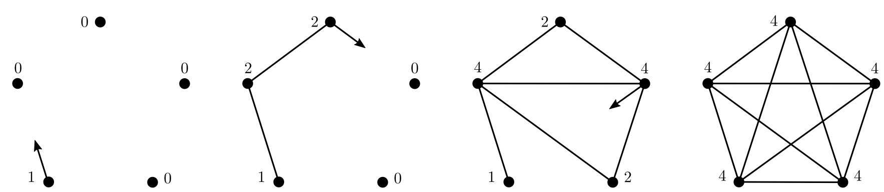


Figure 36: "Traveling" along an Euler cycle in K_5; numbers indicate vertex degrees at each point in "time".

definition of an Euler cycle requires that we end where we began, and so the final edge takes us to v_0, finally increasing its degree by exactly 1 and making it even again. At this point, the degrees of all vertices, including the single vertex from which we began and at which we ended, are all even. Figure 36 illustrates traveling along the edges of K_5 to construct an Euler cycle. $\square$

The above proof only shows that if a graph has an Euler cycle, then all of its vertices must have even degree. It does not, however, show that if all vertices of a (connected) graph have even degrees then it must have an Euler cycle. The proof for this second part of Euler's theorem is more complicated, and can be found in most introductory textbooks on graph theory.

Our proof above might motivate us to think about what happens if the Euler path we are considering is not a cycle. In other words, what happens if we travel along every edge in a graph but do not return to our starting point? Notice that the degree of the starting point v_0 will then remain odd, as will the last vertex which we visited, since we "entered" it but never "exited" it. Can any of the other vertices in the graph have odd degree? No, because all degrees began at 0, and only increased by 2 when they were visited, with the exception of the vertex from which we began the vertex on which we stopped. Therefore, all vertices have even degree with the exception of two, on which an Euler path begins and ends. This proves a second theorem, one about Euler paths:

Theorem 14. *A graph with more than two odd-degree vertices has no Euler path.*

Hamiltonian Paths and Cycles

Until now we have considered paths and cycles that can visit vertices multiple times. What happens if we require that a path visit every vertex exactly one time? Such paths and cycles are called Hamiltonian paths and Hamiltonian cycles, are the subject of much research. Suppose you would like to fly to every major airport in the continental US without visiting any of them more than once. Is this possible? Despite the similarity of this question to questions we considered in discussing Euler paths and cycles, considerably less is known about them. In particular, there seems to be no good way to look at a particular graph and know whether a Hamiltonian path or cycle exists, without trail and error.

Although less is known about Hamiltonian paths than about Euler paths, many things are known. In particular, it is known that all five of the Platonic solids have Hamiltonian cycles. Furthermore, every complete graph K_n also has a Hamiltonian cycle (can you see why?). In 1952, Gabriel Dirac proved that every (simple) graph on n vertices has a Hamiltonian cycle if the degree of every vertex is $n/2$ or greater. Although this theorem guarantees a Hamiltonian cycle under certain conditions, this does not mean that if a graph has a Hamiltonian cycle, then it must satisfy this condition.