7 Symmetry and Group Theory

One of the most important and beautiful themes unifying many areas of modern mathematics is the study of symmetry. Many of us have an intuitive idea of symmetry, and we often think about certain shapes or patterns as being more or less symmetric than others. A square is in some sense "more symmetric" than a rectangle, which in turn is "more symmetric" than an arbitrary four-sided shape. Can we make these ideas precise? Group theory is the mathematical study of symmetry, and explores general ways of studying it in many distinct settings. Group theory ties together many of the diverse topics we have already explored – including sets, cardinality, number theory, isomorphism, and modular arithmetic – illustrating the deep unity of contemporary mathematics.

7.1 Shapes and Symmetries

Many people have an intuitive idea of symmetry. The shapes in Figure 38 appear symmetric, some perhaps more so than others. However, despite our general intuitions about symmetry, it may not be clcear how to make this statement precise. Can it make sense to discuss "how much" symmetry a shape has? Is

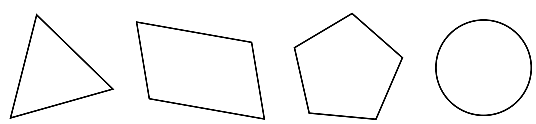
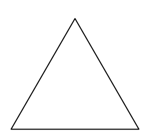


Figure 38: Some symmetric polygons.

there some way to make precise the idea that the regular pentagon is "more symmetric" than the equilateral triangle, or that the circle is "more symmetric" than any regular polygon? In this section we will explore symmetry and the way in which it arises in various contexts with which we are familiar, especially in the geometry of regular polygons (2D) and regular polyhedra (3D), such as the Platonic solids. The study of symmetry is a recurring theme in many disparate areas of modern mathematics, as well as chemistry, physics, and even economics.

To help us explore the idea of symmetry, we begin by considering a single concrete example, the equilateral triangle below. What does it mean for this shape to be symmetric?



Rotation symmetries

An equilateral triangle can be rotated by 120°, 240°, or 360° angles without really changing it. If you were to close your eyes, and a friend rotated the triangle by one of those angles, then after opening your eyes you would not notice that anything had changed. In contrast, if that friend rotated the triangle by 31° or 87°, you would notice that the bottom edge of the triangle is no longer perfectly horizontal.

Many other shapes that are not regular polygons also have rotational symmetries. The shapes illustrated in Figure 39, for example, each have rotational symmetries. The first example can be rotated only 180°, or else 360° or 0°. The third shape can be rotated any integer multiple of 90°. The fourth shape can be rotated any integer multiple of 72°. The fifth shape can be rotated any integer multiple of 60°.

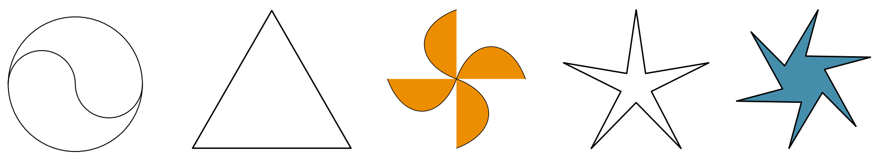


Figure 39: Several shapes with rotational symmetries.

More generally, we say that a shape has **rotational symmetry of order** n if it can be rotated by any multiple of $360°/n$ without changing its appearance. We can imagine constructing other shapes with rotational symmetries of arbitrary order. If the only rotations that leaves a shape unchanged are multiples of 360°, then we say that the shape has only the trivial (order $n = 1$) symmetry.

Mirror symmetries

Another type of symmetry that we can find in two-dimensional geometric shapes is mirror symmetry. More specifically, we can draw a line through some shapes and reflect the shape through this line without changing its appearance. This is called a **mirror symmetry**.

Further consideration of the equilateral triangle (cf. Figure 40) shows that there are actually three distinct mirror lines through which we can reflect the shape without changing its appearance. If we were to reflect the triangle through any other line, the shape as a whole would look different.

Rotational symmetries and mirror symmetries are not exclusive, and the same shape can have symmetries of both kinds. The equilateral triangle clearly has both mirror symmetries and rotational symmetries. Likewise, the fourth shape in Figure 39 has five mirror symmetries, along with many five rotational ones. The shapes in Figure 41, alternatively, have only mirror symmetries but no rotational ones.

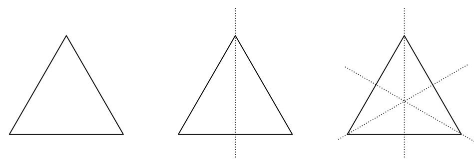


Figure 40: A line can be drawn through a triangle to highlight its symmetry. If the shape is reflected through this line, then we obtain the same equilateral triangle, unmoved.

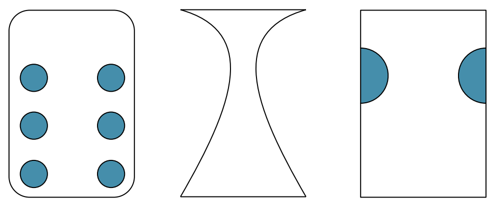


Figure 41: Each of these shapes can be reflected through a vertical line; none of these shapes have any rotational symmetry.

Counting symmetries

One way in which we can quantify the "amount" of symmetry of an object is by counting its number of symmetries. For example, we might count the number of rotational symmetries of an object, along with its mirror reflection symmetries. However, counting the symmetries of a shape can be challenging. It is not immediately clear which symmetries we should count and which, if any, we should not count. To understand why we might not count certain symmetries, consider rotating the equilateral triangle by $120°$, $240°$, and $360°$. Of course the numbers by which we are rotating the triangle are different, and so we might be inclined to count each of them separately. But notice that we can also rotate the triangle by $480°$, $600°$, and $720°$. Should we count those as different symmetries? If we do count them, then what would stop us from counting an infinite number of rotational symmetries for a triangle, or for that matter, any shape?

One way to limit the number of symmetries we count involves coloring, or otherwise labeling, the shape. For example, we can color each edge of the equilateral triangle, as illustrated in Figure 42. Symmetries can then be captured as changes of colors that leave the uncolored shape fixed. Any triangle in either row can be obtained from any other triangle in that row through a rotation; triangles can be obtained from triangles in the other row through reflections.

Using this coloring allows us to count symmetries carefully. If changing the shape in two different ways results in the same coloring, then we should count those two symmetries as the same. For example, rotating the equilateral triangle by $120°$ or $480°$ results in the same coloring, so we count those as the same symmetry. Likewise, rotating the triangle by $0°$ and $360°$ also result in

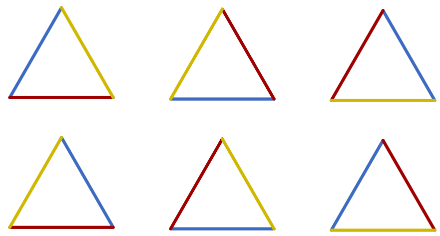


Figure 42: Equilateral triangle with edges colored. Any triangle in either row can be obtained from another triangle in the row through a rotation; triangles can be obtained from triangles in the other row through reflections.

the same coloring, so we count those the same as well. To reduce confusion, we use a number between 0 and 360 (not including 360 itself) to describe the angle of a rotation; thus, we prefer 120° to 480°, despite their equivalence. Likewise, for reasons that will become more clear in the following section, we discussing 0° rotations, or "doing nothing" to 360° rotations, despite their equivalence.

We are therefore left with six symmetries of the triangle – the rotations (0°, 120°, and 240°), and three reflections, one for each of the mirror planes passing through a corner and the center of the triangle. These symmetries can be pictured by how they transform the colored triangle in Figure 42.

Symmetries of the square

A square is in some sense "more symmetric" than a triangle because it has more symmetries. Figure 43 below shows a square with colored edges arranged in different ways. Again you might notice that any two squares in the same row can be obtained from one another through rotations, whereas those in distinct rows can only be obtained from one another through a reflection. Some thought

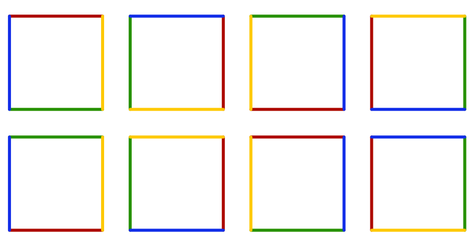


Figure 43: Squares.

will show that there are no other rotations or mirror symmetries, and so these figures represent all eight symmetries of the square.

Although this section is concerned primarily with rotational and mirror symmetries of single objects in two dimensions, other types of symmetries arise in infinite systems and in higher dimensions. We do not consider those symmetries in this section.

Until now we have considered what symmetries are and briefly discussed how to count them. To say that a particular shape is "more symmetric" than another one can be made precise by considering their total number of symmetries. For example, the three shapes in Figure 44 each have a set of four symmetries. However, notice that the first two shapes have the same set of 4 rotational symmetries (0°, 90°, 180°, and 270°), but no mirror symmetries. In contrast, the third shape has 2 rotational symmetries and 2 mirror symmetries. How

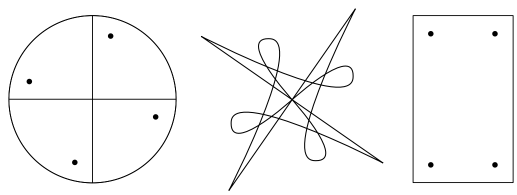


Figure 44: Three shapes, each with 4 symmetries. The first two have 4 rotational symmetries (0°, 90°, 180°, and 270°) and no mirror symmetries. The third has 2 rotational symmetries (0° and 180°), and two mirror symmetries.

can we distinguish between the first and second shapes, on the one hand, and the third shape, on the other? The mathematical development of group theory provides rigorous tools to describe symmetries of shapes. Before consider the actual definition of a *group*, we first consider a more general topic of binary operators.