

7.2 Binary Operators

A precise discussion of symmetry benefits from the development of what mathematicians call a *group*, which is a special kind of set we have not yet explicitly considered. However, before we define a group and explore its properties, we reconsider several familiar sets and some of their most basic features.

Over the last several sections, we have considered many different kinds of sets. We have considered sets of integers (natural numbers, even numbers, odd numbers), sets of rational numbers, sets of vertices, edges, colors, polyhedra and many others. In many of these examples – though certainly not in all of them – we are familiar with rules that tell us how to combine two elements to form another element. For example, if we are dealing with the natural numbers, we might consider the rules of addition, or the rules of multiplication, both of which tell us how to take two elements of \mathbb{N} and combine them to give us a (possibly distinct) third element.

This motivates the following definition.

Definition 26. *Given a set S , a **binary operator** \star is a rule that takes two elements $a, b \in S$ and manipulates them to give us a third, not necessarily distinct, element $a \star b$.*

Although the term binary operator might be new to us, we are already familiar with many examples. As hinted to earlier, the rule for adding two numbers to give us a third number is a binary operator on the set of integers, or on the set of rational numbers, or on the set of real numbers. We can specify a set and binary operator on that set by writing down the set and then the operator: S, \star . For example, we might specify $\mathbb{Z}, +$ or $\mathbb{N}, /$ to denote the integers and addition on them, or else the natural numbers and division.

Binary operators can be defined on arbitrary sets, not only sets of numbers. For example, we might consider a set C of colors, and define a binary operator \bowtie which tells us how to combine two colors to form another color. If $C = \{\text{red, yellow, green, blue, purple}\}$, then we can write $\text{red} \bowtie \text{blue} = \text{purple}$, since we can combine the first two colors to make the third. We might also consider a set of sets, and consider the binary operator \cup , the union, which takes any two sets and “combines” them by giving us their union.

Keep in mind that not every set has a natural binary operator. For example, consider the set $F = \{\text{table, chair, bookshelf, couch, windows, } \dots\}$ of furniture in your house. While this set appears quite natural, there does not appear to be any natural rule for “combining” its elements.

After understanding what a binary operator is, we next look at several properties that a binary operator might or might not possess.

Closure

Some binary operators are such that when we combine two elements from a set, we always get another element in that set.

Definition 27. A binary operator \star is **closed** on a set S if for every $a, b \in S$, $a \star b$ is also an element of S .

Many sets with which we are familiar are closed under particular binary operators, while many are not. Let's consider several examples.

Example 1. \mathbb{N} , $+$, the natural numbers under addition. If a and b are both elements of \mathbb{N} , then their sum $a + b$ is also an element of \mathbb{N} . Therefore, \mathbb{N} is closed under addition.

Example 2. Consider the odd integers under multiplication. If a and b are both odd, then their product $a \times b$ is also odd. Therefore, the set of odd integers is closed under multiplication.

Example 3. The set of odd integers is not closed under addition, since the sum of two odd numbers is not always odd (in fact, it is never odd).

Identity element

For many choices of a set and binary operator, there exists a special element in the set that when “combined” with other elements in the set does not change them. Such an element is called a neutral, or identity, element. When considering addition on the real numbers, for example, the number 0 is unique in that it can be added to any other number and leave that number unchanged.

Definition 28. For a set S and binary operator \star , an **identity element** $e \in S$ is one such that for any element $a \in S$, we have $a \star e = e \star a = a$.

Example 4. Consider addition on the integers. 0 is an identity element, since for all $a \in \mathbb{Z}$ we have $0 + a = a + 0 = a$.

Example 5. Consider multiplication on the natural numbers. 1 is an identity element, since for all $a \in \mathbb{Z}$ we have $1 \times a = a \times 1 = a$.

Example 6. Consider a set whose elements are other sets. The empty set \emptyset is an identity element for the binary operator \cup .

Example 7. Consider the even integers. Under addition there is an identity element (which is 0), but under multiplication there is no identity element (since 1 is not an even number). This illustrates the important point that not all sets and binary operators have an identity element.

Example 8. We earlier considered a set of colors, and a rule for combining them. Such a set and binary operator might have no identity element. Or perhaps there is a clear paint so that combining it with any color leaves that color unchanged.

Inverses

We know from elementary school that for every number x , there is another number (which we often write as $-x$) such that when we add it to x , we get the identity element 0. For example, 5 has an “inverse” -5, and adding them together gives us 0. Such inverses exist not only for numbers under addition, but also for many other choices of sets and binary operators. For some choices of sets and binary operators, for every element there is another element so that combining the two elements always gives us the identity element.

Definition 29. For a set S and binary operator \star , an element $a' \in S$ is called the **inverse** of a if $a \star a' = a' \star a = e$.

This idea generalized the concept of a negative in addition, and the concept of reciprocal we find in multiplication to arbitrary sets and binary operators.

Example 9. Consider addition on the integers. For every integer $a \in \mathbb{Z}$ there exists another element a' such that $a + a' = 0$. We often write this inverse as $-a$, so we have $a + -a = -a + a = e = 0$. Sometimes we refer to these as additive inverses.

Example 10. Consider the rational numbers \mathbb{Q} under multiplication. Most elements in \mathbb{Q} have multiplicative inverses. For example, if $a = 3/5$, then there is another element $a' = 5/3 \in \mathbb{Q}$ so that $a \times a' = a' \times a = e = 1$. However, 0 is in \mathbb{Q} but does not have an inverse under multiplication, as there is no rational number q such that $0 \times q = q \times 0 = 1$. This highlights the point that it is possible for some elements to have inverses while others do not.

Example 11. Consider the set $S = \{1, 3, 5, 7\}$ under multiplication modulo 8; in this case 1 is an identity element, since $a \times 1 = 1 \times a = a$ for any element a . Under multiplication modulo 8, every element in S has an inverse. In fact, each element of S is its own inverse, as $a \times a \equiv 1 \pmod{8}$ for all $a \in S$.

Example 12. Consider the set $S = \mathbb{N} \cup \{0\}$ (the set of all non-negative integers) under addition. The number 0 is an identity element, since for all elements $a \in S$ we have $a + 0 = 0 + a = a$. However, no element except for 0 has an inverse.

Combined, the above examples illustrate that sometimes all elements of a set have an inverse, sometimes almost all elements have an inverse, and sometimes almost none of the elements have an inverse.

Associativity

As described above, binary operators take two elements and combine them to produce a third. However, occasionally we will write things such as $5 + 8 + 2$, in which we have a total of three elements that should be combined, and there exists some ambiguity as to which of the following two procedures we should follow:

1. Add 5 and 8 to obtain 13, and then add 13 with 2 to obtain 15, or else

2. Add 8 and 2 to obtain 10, and then add 10 with 5 to obtain 15.

In this particular case, a small miracle occurs, and the two resulting numbers are the same. The two processes are procedurally different, but bottom line they end with the same result. The reader probably knows from elementary school, though, that there is nothing special about the numbers 5, 8, and 2, and in fact the same small miracle would occur with any three numbers. This motivates the definition of a final important property of binary operators.

Definition 30. A binary operator \star on a set S is called **associative** if for all $a, b, c \in S$ we have

$$a \star (b \star c) = (a \star b) \star c. \quad (83)$$

In elementary school we learned that addition is associative, and so the order in which we add a set of numbers is not important. Likewise, multiplication is also associative, and for any numbers x, y, z we have $x(yz) = (xy)z$. It is for this reason that we can write xyz without bothering to specify whether we intend that x and y be combined first, or whether we intend that y and z be combined first – it doesn't matter.

Although most of us take the associativity of addition and multiplication for granted, we are all familiar with other binary operators which are not associative. For example, if we write $5 - 8 - 2$, the order in which we performing the operations matters, i.e., $(5 - 8) - 2 \neq 5 - (8 - 2)$!

Further complications arise when we mix operators. For example, the expression $5 + 8 \times 2$ can be interpreted in two different ways, making an important difference in the result. Should we add 5 and 8 and then multiply the result by 2 (to obtain 26)? or should we add 5 to the product of 8 and 2 (to obtain 21)? Although there is no “right” answer, certain conventions have evolved, and in general, if parentheses do not indicate otherwise, we first consider multiplications, and only then consider additions. Notice, however, that the famous “order of operations” does not directly address the “correct” interpretation of a statement such as $5/8/2$, which could result in $5/16$ or $5/4$, depending which division we calculate first.

Example 13. Consider the integers under addition. For any integers $a, b, c \in \mathbb{Z}$ we have $a + (b + c) = (a + b) + c$.

Example 14. Consider the real numbers under multiplication. For any real numbers $a, b, c \in \mathbb{R}$ we have $a \times (b \times c) = (a \times b) \times c$.

Example 15. Consider positive rational numbers under division. For positive rational numbers a, b, c it is generally the case that $a/(b/c) \neq (a/b)/c$, and so division is not associative. The same is true for subtraction.

Example 16. Consider a set of colors and a binary operator of mixing them. If you take three colors and mix them, the order in which you do will change the intermediate colors observed, but will not change the final color.

Most binary operators we consider will be associative.

Commutativity

Binary operators are rules for taking two elements from a set and combining them to produce something. Addition, subtraction, multiplication, and division are all binary operators with which we are familiar from grade school. For all real numbers a and b , it is always true that $a + b = b + a$ and that $a \times b = b \times a$. Of course we know that this is not the case for subtraction and multiplication, since for almost all a and b , $a - b \neq b - a$ and $\frac{a}{b} \neq \frac{b}{a}$. These motivate the following definition:

Definition 31. A binary operator \star on a set S is called **commutative** if for all $a, b \in S$ we have

$$a \star b = b \star a. \tag{84}$$

Example 17. Consider the binary operator of exponentiation (on the integers, rationals, or reals). For almost all a and b we have $a^b \neq b^a$.

Example 18. We noted earlier that binary operators can act not only on numbers, but also on arbitrary elements, such as colors or other sets. Here we consider a set of geometric transformations which we can combine:

$$S = \{ \text{rotate } 0^\circ, \text{rotate } 90^\circ, \text{rotate } 180^\circ, \text{rotate } 270^\circ, \\ \text{vertical mirror, horizontal mirror} \}$$

We can consider the binary operator of combining these geometric transformations by doing one and then doing the other. For example, consider reflecting a picture about a vertical mirror through its center and then rotating it by 90° , or rotating it by 180° and then reflecting it by about a horizontal mirror through its center. Spend a few minutes thinking about whether or not S is closed under this binary operator (HINT: it is not!).

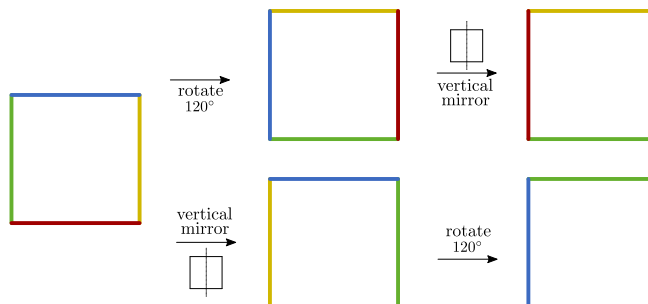


Figure 45: Square changed with a 90° rotation and with a reflection through a vertical mirror; the order in which these two operations are performed matters.

Of course, if you begin with an arbitrary picture, rotating it by 90° will give you a different result than flipping it about a vertical mirror. But what happens when we apply two of these transformations, but in different orders? Perform

the following thought experiment. Take any two elements a and b from S and combine them, first performing a followed by b , and then performing b followed by a . Do you obtain the same result? Try performing this exercise for several choices of a and b .

If you spent several minutes performing the thought experiment, you will probably notice that in general $a \star b \neq b \star a$. In other words, the operation of combining geometric transformations is not commutative. Figure 45 illustrates what happens when we apply a 90° rotation and a reflection through a vertical mirror to a colored square in two different orders. Although we are performing the same two operations on the square, the final product depends on the order in which we perform them.