7.4 Symmetry Groups of Shapes

One of the primary applications of group theory is the study of symmetries of shapes of different kinds. Symmetries of shapes form groups, and this section will explore many such examples, including those associated with regular polygons and polyhedra.

Cyclic Groups

Consider the set of rotations of an equilateral triangle that we considered before. We have the set:

$$S = \{\ \text{rotate } 0°, \text{rotate } 120°, \text{rotate } 240°\ \}, \tag{85}$$

which as we have seen before forms a group under the binary operation defined by performing one rotation and then another. For reasons that will become clear soon, from now onwards we will call this group C_3. Likewise, C_4 will be the group:

$$C_4 = \{\ \text{rotate } 0°, \text{rotate } 90°, \text{rotate } 180°, \text{rotate } 270°\ \}. \tag{86}$$

One thing we might notice about these two groups is that all elements of the group can be obtained by taking one element of the set, and combining it different numbers of times. For example, let us use r to denote the rotation by 90°. We can then rewrite C_4 as:

$$C_4 = \{\ r^0, r^1, r^2, r^3\ \}, \tag{87}$$

where powers of r indicate performing the same geometric operation (in this case rotations by 90°) multiple times. If we s to denote a rotation by 120°, then we can likewise describe C_3 as the set $\{s^0, s^1, s^2\}$.

Both of these examples illustrate the possibility of "generating" certain groups by using a single element of the group, and combining it different numbers of times. We have a special name for such groups:

Definition 34. *A* **cyclic group** *is a group that can be "generated" by combining a single element of the group multiple times. A cyclic group with n elements is commonly named C_n.*

Figure 48 illustrates several shapes with symmetry groups that are cyclic.

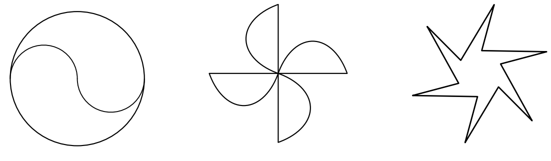

Figure 48: Shapes with associated symmetry groups C_2, C_4, and C_6.

The examples above might lead us to wonder whether all symmetry group can in fact be generated by repeatedly combining a single element. Is every symmetry group in fact cyclic? Simple consideration will show us that this is not the case.

Dihedral Groups

Let us reconsider, for example, the set of all symmetries of a square. In addition to four rotational symmetries $(0°, 90°, 180°, 270°)$, the square also has four mirror reflection symmetries; the effects of applying these symmetries to a colored square can be seen in Figure 50. If the first square is identified with the identity

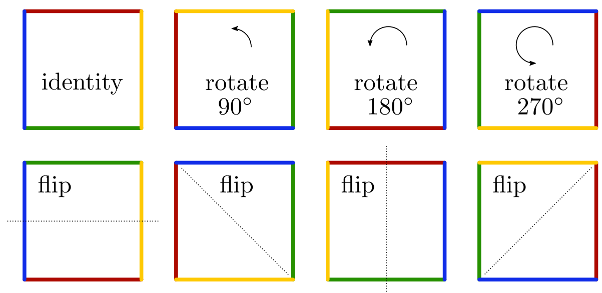

Figure 49: A single colored square transformed by rotations and mirror reflections; the set of all n rotation symmetries and n mirror reflection symmetries of a regular polygon with n sides make up the symmetry group of that polygon.

element ($0°$ rotation), then squares in the first row illustrate rotations, and all squares in the second row illustrate mirror reflections. It turns out that this set of rotations and reflections satisfy all criteria to form a group.

Knowing that the symmetries of a square constitute a group, we might wonder whether this group is cyclic. In other words, can all of these symmetries be obtained by combining a single element multiple times? In short, the answer is no. To help understand why this is, consider that repeating a mirror reflection returns a shape to its original position; i.e., every mirror reflection is its own inverse. Therefore, a single mirror reflection cannot possible generate any elements aside from the identity and itself. Likewise, a single rotation combined with itself many times could never produce a mirror reflection. To see why, notice that all elements in the top row of Figure 50 have the same "orientation". Specifically, red, yellow, green, blue all appear in the same order (clockwise). Rotations never change the orientation of a shape. In the bottom row, the four colors appear in a reversed order, which happens under any mirror reflection symmetry. In short, the symmetry group of a square is not cyclic.

Definition 35. *A* **dihedral group** *is a group that can be "generated" by combining a rotation symmetry and a mirror reflection multiple times. A dihedral group with n rotational and n mirror symmetries is commonly named D_n.*

Dihedral groups are often associated with regular polygons. In particular, the set of symmetries of every regular polygon with n sides forms the dihedral group D_n. Since this group contains n rotations and n reflection symmetries, the order of D_n is always $2n$.

Symmetry Groups of the Platonic Solids

The Platonic solids have symmetry groups that are even more complicated than either the cyclic or dihedral groups. One way to understand this is through consideration of their rotational symmetries. Until now all symmetry groups associated with shapes have a single axis of rotation. In both the cyclic and dihedral group, all rotational symmetries can be obtained by repeating a single rotation multiple times. This is not the case, however, for three-dimensional shapes including the Platonic solids.

[Notes here are incomplete.] However, we briefly consider one example. The cube has three different kinds of rotational symmetries.

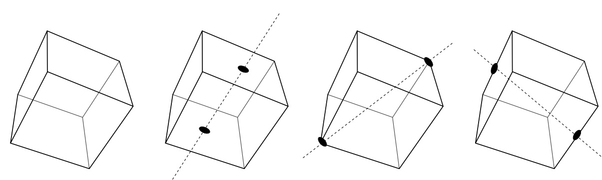

Figure 50: A cube and three different kinds of rotational symmetry axes.

1. We can draw lines through centers of opposite faces, and rotate the cube by multiples of 90° about these; there are three such pairs of faces, and hence three such axes of rotation.

2. We can also draw lines through opposite corners, and rotate the cube by multiples of 120° about these lines; there are four pairs of corners, and hence four such axes of rotation.

3. We can also draw lines through centers of opposite edges, and rotate the cube by multiples of 180° about these lines; there are six pairs of edges, and hence six such axes of rotation.

By themselves, these symmetries do not form a group, since in general combining these symmetries produce another symmetry not in this list. However, by combining these symmetries we can form a group. If we ignore any symmetries, the set of symmetries of a cube has 24 elements; if we include mirror reflections, the group of symmetries has 48 elements. [Notes here are incomplete.]