# The Topology and Combinatorics of Soccer Balls

*When mathematicians think about soccer balls, the number of possible designs quickly multiplies*

Dieter Kotschick

With the arrival of the quadrennial soccer World Cup this summer, more than a billion people around the world are finding their television and computer screens filled with depictions of soccer balls. In Germany, where the World Cup matches are being played, soccer balls are turning up on all kinds of merchandise, much of it having nothing to do with soccer.

Although a soccer ball can be put together in many different ways, there is one design so ubiquitous that it has become iconic. This standard soccer ball is stitched or glued together from 32 polygons, 12 of them five-sided and 20 six-sided, arranged in such a way that every pentagon is surrounded by hexagons. Postmodern paint jobs notwithstanding, the traditional way to color such a ball is to paint the pentagons black and the hexagons white. This color scheme was reportedly introduced for the World Cup in 1970 to enhance the visibility of the ball on television, although the design itself is older.

Most people associate the soccer-ball image with hours spent on the field or the sidelines, or perhaps just with advertisements for sport merchandise. But to a mathematician, a soccer ball is an intriguing puzzle. Why does it look the way it does? Are there other ways of putting it together? Could the pentagons and hexagons be arranged differently? Could other polygons be used instead of pentagons and hexagons? These questions can be tackled using the language of mathematics—in particular geometry, group theory, topology and graph theory. Each of these subjects provides concepts and a natural context for phrasing questions such as those about the design of soccer balls, and sometimes for answering them as well.

An important aspect of the application of mathematics is that different ways of making mathematical sense of everyday questions lead to different answers. This may come as a bit of a surprise to readers who are used to schoolbook problems that have only one right answer. Properly framing questions is just as important a part of the art of mathematics as answering them. Moreover, a genuine mathematical exploration of an open-ended question does not stop with finding "the answer" (if there is one), but involves understanding why the answer is what it is, and how it changes when the underlying assumptions are modified. The questions posed by the design of soccer balls provide a wonderful illustration of this process.

**Soccer Balls and Fullerenes**

Mathematicians like to begin by defining their terms. What, then, is a soccer ball? An official soccer ball, to be approved by the Fédération Internationale de Football Association (FIFA), must be a sphere with a circumference between 68 and 70 centimeters, with at most a 1.5 percent deviation from sphericity when inflated to a pressure of 0.8 atmospheres.

Alas, such a definition says nothing about how the ball is put together, and is therefore not suitable for a mathematical exploration of the design. A better definition is that a soccer ball is approximately a sphere made of polygons, or what mathematicians call a *spherical polyhedron*. The places where the polygons come together—the vertices and edges of the polyhedron—trace out a map on the sphere, which is called a *graph.* (Such a graph has nothing to do with graphs of functions. The word has two completely different mathematical meanings.) Examined from the perspective of graph theory, the standard soccer ball has three important properties:

(1) it is a polyhedron that consists only of pentagons and hexagons;
(2) the sides of each pentagon meet only hexagons; and
(3) the sides of each hexagon alternately meet pentagons and hexagons.

As a starting point, then, we can *define* a soccer ball to be any spherical polyhedron with properties (1), (2) and (3). If the pentagons are painted black and the hexagons are painted white, then the definition does capture the iconic image, though it does not determine it uniquely.

This definition places the problem of soccer ball design into the context of graph theory and topology. Topology, often described as "rubber-sheet geometry," is the branch of mathematics that studies properties of objects that are unchanged by continuous deformations, such as the inflation of a soccer ball. For the purposes of topology, it doesn't matter how long the edges of a polyhedron are, or whether we are dealing with a round polyhedron or one with flat sides.

I first encountered the above definition in 1983, in a problem posed in the Bundeswettbewerb Mathematik, a German mathematics competition for high school students. The problem was: Given properties (1)–(3), determine how many

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Sabrina Sarnitz/GmbH

**Figure 1. Twelve pentagons and 20 hexagons form a figure known to mathematicians as a truncated icosahedron, to chemists as the buckminsterfullerene molecule—and to nearly everybody else as the standard soccer ball. As this summer's World Cup competition approached, a soccer ball–shaped information pavilion toured the German cities preparing to host World Cup matches. Here the "football globe" makes a stop in Leipzig. The iconic black and white soccer ball is also an intriguing puzzle amenable to mathematical analysis. Other soccer balls that may never be seen on the playing field also offer interesting solutions to the mathematical questions posed by the standard design.**

pentagons and hexagons a soccer ball is made of. Thinking about this problem at the time, I assumed that the ball is a convex polyhedron in space made up of regular polygons. This geometric assumption, together with rules (1), (2) and (3), implies that there are 12 pentagons and 20 hexagons. Moreover, there is a unique way of putting them together, giving rise to the iconic standard soccer ball. Without the geometric assumption, the graph-theory problem has infinitely many other solutions, which have larger numbers of pentagons and hexagons.

I began thinking about this problem again after I was invited to give a lecture at a prize ceremony for the same competition in 2001. Eventually, one of my postdoctoral fellows, Volker Braungardt, and I found a way to characterize all the solutions, a characterization that I will describe below.

Interestingly, a related problem arose in chemistry in the 1980s after the 60-atom carbon molecule, called the buckminsterfullerene or "buckyball," was discovered. The spatial shape of this $C_{60}$ molecule is identical to the standard soccer-ball polyhedron consisting of 12 pentagons and 20 hexagons, with the 60 carbon atoms placed at the vertices and the edges corresponding to chemical bonds. The discovery of the buckyball, which was honored by the 1996 Nobel Prize for chemistry, created enormous interest in a class of carbon molecules called fullerenes, which satisfy assumption (1) above together with a further condition:

(3′) precisely three edges meet at every vertex.

This property is forced by the chemical bonding properties of carbon. In addition, assumption (2) is sometimes imposed to define a restricted class of fullerenes. Having disjoint pentagons is expected to be related to the chemical stability of fullerenes. There are infinitely many fullerene polyhedra—$C_{60}$ was merely the first one discovered as an actual molecule—and it is quite remarkable that the two infinite families of polyhedra, the soccer balls and the fullerenes, have only the standard soccer ball in common. Thus (1)–(3) together with (3′) give a unique description of the standard soccer ball without imposing geometric assumptions. (Assumptions like regularity in fact imply condition (3′).)

To see that this is so requires a brief excursion into properties of polyhedra, starting with a beautiful formula discovered by the Swiss mathematician Leonhard Euler in the 18th century. Euler's formula *(see "Euler's formula," next page)*, a basic tool in graph theory and topology, says that in any spherical polyhedron, the number of vertices, $v$, minus the number of edges, $e$, plus the number of faces, $f$, equals 2:

$$v - e + f = 2$$

Let's apply Euler's formula to a polyhedron consisting of $b$ black pentagons and $w$ white hexagons. The total number $f$ of faces is $b + w$. In all, the pentagons have $5b$ edges, because there are 5 edges per pentagon and $b$ pentagons in


**Figure 2. Soccer-ball design over the years has been driven by the demand for a round ball that holds its shape and by the technology available. Eight panels of vulcanized rubber were glued together to create the ball at left, used in the earliest soccer championship in the United States in 1863. The leather ball at center, used in the 1950 World Cup, has a design typical of its era. The small number of large, irregularly shaped flat pieces adversely affected its roundness. Thanks to improvements in materials and manufacturing, curved pieces in more complicated shapes can now be used. This year's World Cup ball *(right)* is made from 14 synthetic-leather panels cut in intricately curved shapes. In graph-theory terms, this ball is a truncated octahedron. (Historic photographs courtesy of Jack Huckel, National Soccer Hall of Fame; right photograph courtesy of firosportfoto.de.)**

all. Similarly, the hexagons have a total of $6w$ edges. Adding these two numbers should give the total number of edges—except that I have counted each edge twice because each edge lies in two different faces. To compensate I divide by 2, and hence the number of edges is:

$$e = (1/2)(5b + 6w)$$

Finally, to count the number of vertices, I note that the pentagons have $5b$ vertices in all and the hexagons have $6w$ vertices. In the case of a fullerene, assumption (3′) says that each vertex belongs to three different faces. Thus if I compute $5b + 6w$, I have counted each vertex exactly three times, and hence I must divide by 3 to compensate:

$$v = (1/3)(5b + 6w)$$

Substituting these values for $f$, $e$ and $v$ into Euler's formula, I find that the terms involving $w$ cancel out, and the formula reduces to $b = 12$. Every fullerene, therefore, contains exactly 12 pentagons! However, there is no *a priori* limit to the number of hexagons, $w$, and therefore no limit on the number of vertices. (This is implicit in the title of a 1997 article on fullerenes in *American Scientist*: "Fullerene Nanotubes: $C_{1,000,000}$ and Beyond.") If I impose the additional condition (2), then I can show that the number of hexagons has to be at least 20. The standard soccer ball or buckyball realizes this minimum value, for which the number $v$ of vertices equals 60, corresponding to the 60 atoms in the $C_{60}$ molecule. However, it can be shown that there are indeed infinitely many other mathematical possibilities for fullerene-shaped polyhedra. Which of these correspond to actual molecules is a subject of research in chemistry.

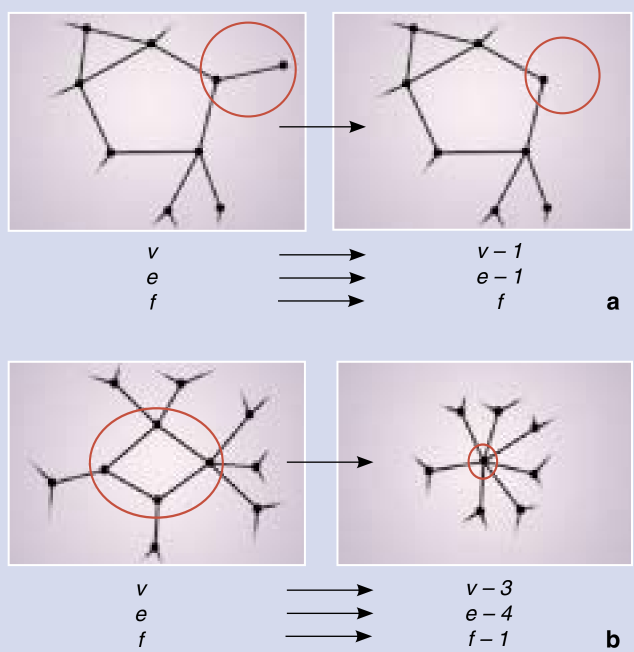
For soccer balls, we are allowed to use only assumptions (1)–(3), but not (3′), the carbon chemist's requirement that three edges meet at every vertex. In this case the number of faces meeting at a vertex is not fixed, but this number is at least 3. Therefore, the equation $v = (1/3)(5b + 6w)$ becomes an *inequality*: $v \leq (1/3)(5b + 6w)$. Substi-

## Euler's formula

Any non-empty connected finite graph on the sphere satisfies Euler's formula $v - e + f = 2$. Here $v$ and $e$ are the numbers of vertices and edges, and $f$ is the number of regions into which the sphere is divided. A proof of Euler's formula proceeds by repeatedly simplifying the graph by the following two operations:

The first operation consists of deleting any vertex that meets only one edge, and in addition deleting the edge that meets it *(a)*. This operation does not change the number of regions, while it decreases both $v$ and $e$ by 1. The second operation consists of collapsing a region to a single vertex, together with all the edges and vertices on its boundary *(b)*. If the collapsed region had $k$ vertices on its boundary, then this collapsing reduces $v$ by $k$–1, reduces $e$ by $k$ and reduces $f$ by 1. Thus $v - e + f$ is not changed by either of the two operations.

A finite iteration of these two simplifications reduces any graph to a graph with only one vertex and no edges. Then there is one region, and $v - e + f = 1 - 0 + 1 = 2$.



tuting into Euler's formula, the terms involving $w$ again cancel out, leaving the inequality $b \geq 12$. Thus every soccer ball contains at least 12 pentagons, but, unlike a fullerene, may well contain more.

Also unlike fullerenes, soccer balls have a precise relation between the number of pentagons and the number of hexagons. Counting the number of edges along which pentagons and hexagons meet, condition (2) says that all edges of pentagons are also edges of hexagons, and condition (3) says that exactly half of the edges of hexagons are also edges of pentagons. Hence $(1/2)(6w) = 5b$, or $3w = 5b$. Because $b \geq 12$, $w$ is at least 20. These minimal values are realized by the standard soccer ball, and the realization is combinatorially unique because of conditions (2) and (3). But there are also infinitely many other numerical solutions, and the problem arises whether these non-minimal numerical solutions correspond to soccer-ball polyhedra. It turns out that they do, as we'll see shortly, so that there is indeed an infinite collection of soccer balls.

Thus we see that there are infinitely many fullerenes (satisfying assumptions (1), (2) and (3′)) and infinitely many soccer balls (satisfying (1), (2) and (3)). However, if we combine the two definitions, there is only one possibility! For a fullerene, $b = 12$, and for a soccer ball, $5b = 3w$. Consequently, for a soccer ball to also be a fullerene, we must conclude that $5 \times 12 = 3w$, or $w = 20$. Any soccer ball that is also a fullerene must therefore have 12 pentagons and 20 hexagons. It is known that there are 1,812 distinct fullerenes with 12 pentagons and 20 hexagons, but 1,811 of them have adjacent pentagons somewhere and are therefore not soccer balls, because they violate condition (2). The standard soccer ball is the only one with no adjacent pentagons.

**New Soccer Balls from Old**

Leaving behind chemistry and fullerene graphs, let us now consider the crucial question: What other, nonstandard, soccer balls are there, with more than three faces meeting at some vertex, and how can we understand them? It turns out that we can generate infinite sequences of different soccer balls by a topological construction called a *branched covering*. You can visualize this by imagining the standard soccer-ball pattern superimposed on the surface of the Earth and aligned so that there is one vertex at the North Pole and one vertex at the South Pole. Now distort the pattern so that one of the zigzag paths along edges from pole to pole straightens out and lies on a meridian, say the prime meridian of zero geographical longitude *(see Figure 4b)*. It is all right to distort the graph, because we are doing "rubber-sheet geometry."

Next, imagine slicing the Earth open along the prime meridian. Shrink the sliced-open coat of the Earth in the east-west direction, holding the poles fixed, until the coat covers exactly half the sphere, say the Western Hemisphere. Finally, take a copy of this shrunken coat and rotate it around the north-south axis until it covers the Eastern Hemisphere. Remarkably, the two pieces can be sewn together, giving the sphere a new structure of a soccer ball with twice as many pentagons and hexagons as before. The reason is that at each of the two seams running between the North and South Poles, the two sides of the seam are indistinguishable from the two sides of the cut we made in our original

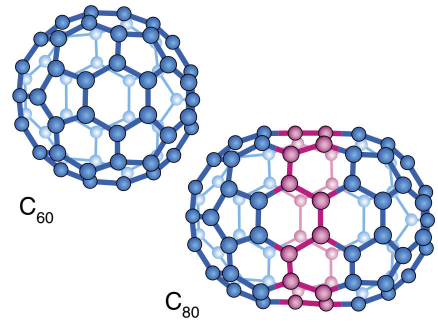



**Figure 3. Fullerenes are large carbon molecules whose shapes are made up of pentagons and hexagons that meet three at a time, in such a way that no two pentagons are adjacent. Every fullerene contains exactly 12 pentagons, but there is no limit to the number of hexagons. The simplest fullerene molecule, $C_{60}$, has the iconic soccer-ball shape. Other fullerenes, such as $C_{80}$, have been made in the laboratory. Mathematically, the combinatorics of fullerenes is an application of Euler's formula.**

soccer ball. Therefore, the two pieces fit together perfectly, in such a way that the adjacency conditions (2) and (3) are preserved. (See Figure 4 for step-by-step illustrations of this construction.)

The new soccer ball constructed in this way is called a *two-fold branched covering* of the original one, and the poles are called *branch points*. The new ball looks the same as the old one (from the topological or rubber-sheet geometry point of view), except at the branch points. There are now six faces (instead of three) meeting at those two vertices, and there are 116 other vertices (the 58 vertices that weren't pinned at the poles, plus their duplicates), with three faces meeting at each of them.

There is a straightforward modification we can make to this construction. Instead of taking two-fold coverings,

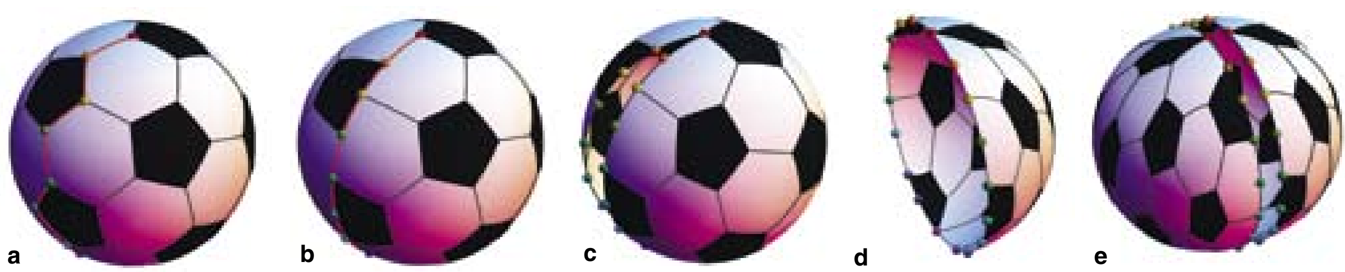



**Figure 4. New soccer balls can be made from existing ones by a mathematical construction called a branched covering. First one chooses a seam of the old soccer ball along the edges of polygons *(a)*. This seam is straightened out and sliced open *(b, c)*. The whole surface of the soccer ball is shrunk to cover only a hemisphere *(d)*. A second copy of this hemisphere is rotated around and stitched to the first *(e, f)*. This builds a new soccer ball, which can be deformed as in *g*. Conforming to the definition of a soccer ball, black faces in the new ball are adjacent to only white faces (faces that meet only at vertices are not considered adjacent), and white faces have an alternating sequence of white and black faces around their edges. (Soccer-ball images in Figures 4, 5, 6, 9 and 11 were calculated and created by Michael Trott using Mathematica.)**

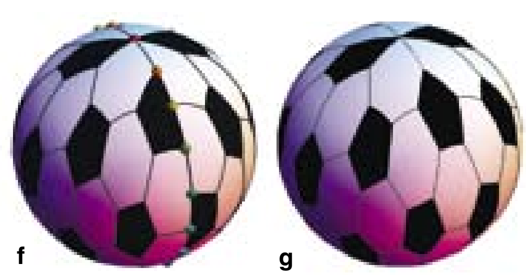


**Figure 5. Infinitely many soccer balls can be constructed by the method used in Figure 4. For example, an eight-fold branched covering of the standard soccer ball can be built by using eight copies of the sliced-open coat of the standard ball to create a soccer ball with 96 pentagons and 160 hexagons. The eight pieces fit together like sections of an orange. The author and his collaborator Volker Braungardt have proved that every soccer ball is a branched covering of the standard one.**

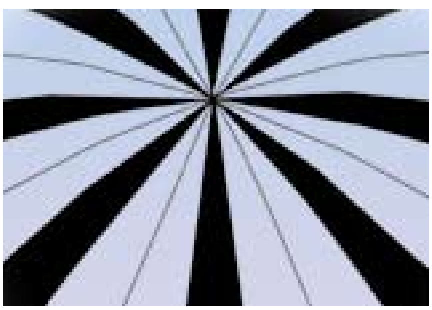


**Figure 6. The proof that branched coverings produce all soccer balls depends on an analysis of the sequence of colors around any vertex. Because at least one of the edges meeting at each vertex bounds a pentagon *(black)*, there is no vertex where only hexagons *(white)* meet. The sequence of faces around a vertex is always black-white-white, black-white-white, and closes up after a number of faces that is a multiple of three.**

we can take $d$-fold branched coverings for any positive integer $d$. Instead of shrinking the sphere halfway, we imagine an orange, made up of $d$ orange sections, and for each section we shrink a copy of the coat of the sphere so that it fits precisely over the section. Once again the different pieces fit together along the seams *(see Figure 5)*. For all of this it is important that we think of soccer balls as combinatorial or topological—not geometric—objects, so that the polygons can be distorted arbitrarily.

At this point you might think that there could be many more examples of soccer balls, perhaps generated from the standard one by other modifications, or perhaps sporadic examples having no apparent connection to the standard soccer ball. But this is not the case! Braungardt and I proved that every soccer ball is in fact a suitable branched covering of the standard one (possibly with slightly more complicated branching than was discussed above).

The proof involved an interesting interplay between the local structure of soccer balls around each vertex and the global structure of branched coverings. Consider any vertex of any soccer ball *(see Figure 6)*. For every face meeting this vertex, there are two consecutive edges that meet there. Because at least one of those two edges bounds a pentagon, by condition (3), there is no vertex where only hexagons meet. Thus at every vertex there is a pentagon. Its sides meet hexagons, and the sides of the hexagons alternately meet pentagons and hexagons. This condition can be met only if the faces are ordered around the vertex in the sequence black, white, white, black, white, white, etc. (Remember that the pentagons are black.) In order for the pattern to close up around the vertex, the number of faces that meet at this vertex must be a multiple of 3. This means that locally, around any vertex, the structure looks just like that of a branched covering of the standard soccer ball around a branch point. Covering space theory—the part of topology that investigates relations between spaces that look locally alike—then enabled us to prove that any soccer ball is in fact a branched covering of the standard one.

**Beyond Pentagons and Hexagons**

To mathematicians, generalization is second nature. Even after something has been proved, it may not be apparent exactly why it is true. Testing the argument in slightly different situations while probing generalizations is an important part of really understanding it, and seeing which of the assumptions used are essential, and which can be dispensed with.

A quick look at the arguments above reveals that there is very little in the analysis of soccer balls that depends on their being made from pentagons and hexagons. So it is natural to define "generalized soccer balls" allowing other kinds of polygons. Imagining that we again color the faces black and white, we assume that the black faces have $k$ edges, and the white faces have $l$ edges each. For conventional soccer balls, $k$ equals 5, and $l$ equals 6. As before, the edges of black faces are required to meet only edges of white faces, and the edges of the white faces alternately meet edges of black and white faces. The alternation of colors forces $l$ to be an even number.

Going one step further in this process of generalization, we can require that every $n$th edge of a white face meets a black face, and all its other edges meet white faces. This forces $l$ to be a multiple of $n$; that is, $l = m \times n$ for some integer $m$. Of course we still require that the edges of black faces meet only white faces. Let us call such a polyhedron a *generalized soccer ball*. Thus the pattern of a generalized soccer ball is described by the three integers $(k, m, n)$, where $k$ is the number of sides in a black face, $l = m \times n$ is the number of sides in a white face, and every $n$th side of a white face meets a black face. The first question we must ask is: Which combinations of $k$, $m$ and $n$ are actually possible for a generalized soccer ball? It turns out that the answer to this question is closely related to the regular polyhedra.

**Regular Polyhedra**

Ancient Greek mathematicians and philosophers were fascinated by the regular polyhedra, also known as *Platonic solids*, attributing to them many mystical properties. The Platonic solids are polyhedra with the greatest possible degree of symmetry: All their faces are equilateral polygons with the same number of sides, and the same number of faces meet at every vertex. Euclid proved in his *Elements* that there are only five such polyhedra: the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron *(see Figure 7)*.

Although Euclid used the geometric definition of Platonic solids, assuming all the polygons to be regular, modern mathematicians know that the argument does not depend on the geometry. In fact, a topological argument using only Euler's formula shows that there are no possibilities other than the five shown in Figure 7.

Each Platonic solid can be described by two numbers: the number $K$ of vertices in each face and the number $M$ of faces meeting at each vertex. If $f$ is the number of faces, then the total number of edges is $e = (1/2)K \times f$, and the number of vertices is $v = (1/M)K \times f$. Substituting these values in Euler's formula $f - v + e = 2$, we find that elementary algebra leads to the equation:

$$\frac{1}{Kf} + \frac{1}{4} = \frac{1}{2K} + \frac{1}{2M}$$

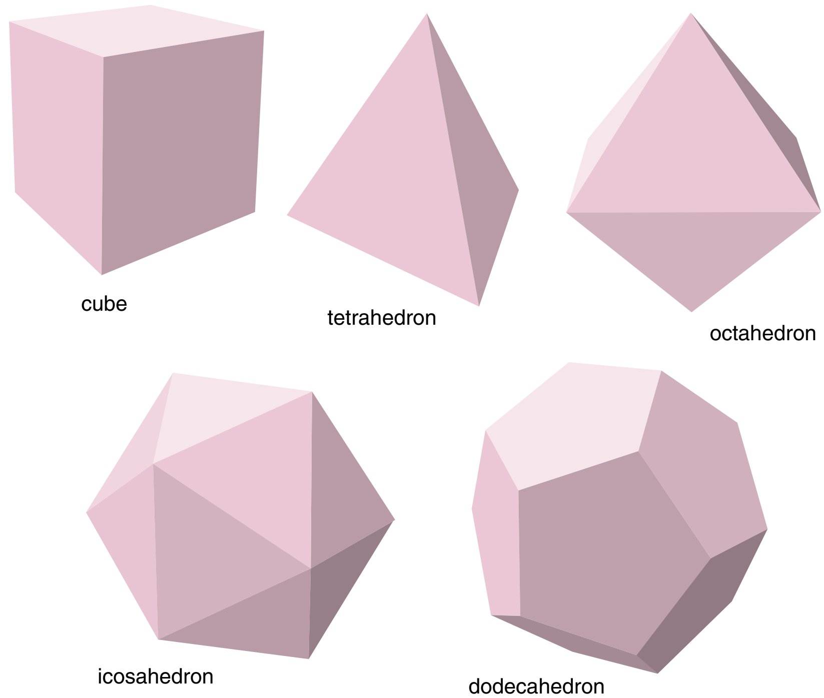



**Figure 7. The five basic Platonic solids shown here have been known since antiquity. Examples of all generalized soccer-ball patterns can be generated by altering Platonic solids.**

The possible solutions can be determined quite easily. The complete list of possible values for the pairs *(K, M)* is:

(3, 3) for the tetrahedron
(4, 3) and (3, 4) for the cube and the octahedron
(5, 3) and (3, 5) for the dodecahedron and the icosahedron.

Strictly speaking, this is only the list of genuine polyhedra satisfying the above equation. The equation does have other solutions in positive integers. These solutions correspond to so-called *degenerate Platonic solid*s, which are not bona fide polyhedra. One family of these degenerate polyhedra has $K$=2 and $M$ arbitrary, and the other has $M$=2 and $K$ arbitrary. The first case can be thought of as a beach ball that is a sphere divided into $M$ sections in the manner of a citrus fruit.

**Finding Generalized Soccer Balls**

The Platonic solids give rise to generalized soccer balls by a procedure known as *truncation*. Suppose we take a sharp knife and slice off each of the corners of an icosahedron. At each of the 12 vertices of the icosahedron, five faces come together at a point. When we slice off each vertex, we get a small pentagon, with one side bordering each of the faces that used to meet at that vertex. At the same time, we change the shape of the 20 triangles that make up the faces of the icosahedron. By cutting off the corners of the triangles, we turn them into hexagons. The sides of the hexagons are of two kinds, which occur alternately: the remnants of the sides of the original triangular faces of the icosahedron, and the new sides produced by lopping off the corners. The first kind of side borders another hexagon, and the second kind touches a pentagon. In fact, the polyhedron we have obtained is nothing but the standard soccer ball. Mathematicians call it the *truncated icosahedron*.

The same truncation procedure can be applied to the other Platonic solids. For example, the truncated tetrahedron consists of triangles and hexagons, such that the sides of the triangles meet only hexagons, while the sides of the hexagons alternately meet triangles and hexagons. This is a generalized soccer ball with $k$=3, $m$=3, $n$=2 (and $l = m \times n = 6$). The truncated icosahedron gives values for $k$, $m$ and $n$ of 5, 3 and 2. The remaining truncations give $(k, m, n)$ = (4, 3, 2) for the octahedron, (3, 4, 2) for the cube, and (3, 5, 2) for the dodecahedron. In addition, we can truncate beach balls to obtain generalized soccer balls with $(k, m, n)$ = ($k$, 2, 2), where $k$ can be any integer greater than 2.

Are these the only possibilities for generalized soccer ball patterns, or are there others? Again, we can answer this question by using Euler's formula, $f - e + v = 2$. Just as we did for the Platonic solids, we can express the number of faces, edges and vertices in terms of our basic data. Here this is the number $b$ of black faces, the number $w$ of white faces, and the parameters $k$, $m$ and $n$. Now, because the number of faces meeting at a vertex is not fixed, we do not obtain an equation, but an inequality expressing the fact that the number of faces meeting at each vertex is at least 3. The result is a constraint on $k$, $m$ and $n$ that can be put in the following form:

$$\frac{1}{kb} + \frac{n+1}{12} \le \frac{1}{2k} + \frac{1}{2m}$$

This may look complicated, but it can easily be analyzed, just like the equation leading to the Platonic solids. It is not hard to show that $n$ can be at most equal to 6, because otherwise the left-hand side would be greater than the right-hand side. With a little more effort, it is possible to compile a complete list of all the possible solutions in integers $k$, $m$ and $n$.

Alas, the story does not end there. There are some triples, such as $(k, m, n)$ = (4, 4, 1), that satisfy the inequality for suitable values of $b$ but do not arise from generalized soccer balls. However, Braungardt and I were able to determine the values of $(k, m, n)$ that do have realizations as soccer balls; these are shown in the table in Figure 9, where we also illustrate the smallest realizations for a few types. Notice that all of the ones with $n$=2 come from truncations of Platonic solids.

The polyhedra listed here have various interesting properties, of which I'll mention just one. Besides entry 10 in this table, which is of course the standard soccer ball, the table contains three

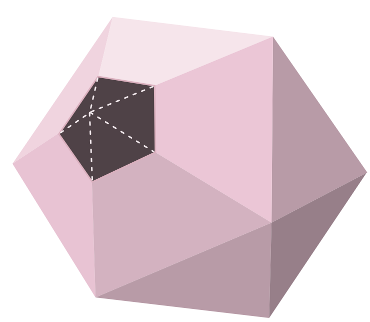
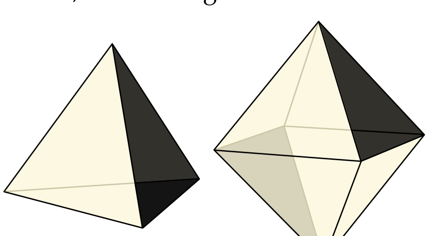


**Figure 8. Chopping off corners, or truncation, converts any Platonic solid into a generalized soccer ball. In particular, the standard soccer ball is a truncated icosahedron. After truncation, the 20 triangular faces of the icosahedron become hexagons; the 12 vertices, as shown here, turn into pentagons.**

| type | $k$ | $m$ | $n$ | minimal realization | $b$ | $w$ |
|---|---|---|---|---|---|---|
| **1** | 3 | 3 | 1 | octahedron | 4 | 4 |
| **2** | 3 | 4 | 1 | cuboctahedron | 8 | 6 |
| **3** | 4 | 3 | 1 | cuboctahedron | 6 | 8 |
| **4** | 3 | 5 | 1 | icosidodecahedron | 20 | 12 |
| **5** | 5 | 3 | 1 | icosidodecahedron | 12 | 20 |
| **6** | 3 | 3 | 2 | truncated tetrahedron | 4 | 4 |
| **7** | 3 | 4 | 2 | truncated cube | 8 | 6 |
| **8** | 4 | 3 | 2 | truncated octahedron | 6 | 8 |
| **9** | 3 | 5 | 2 | truncated dodecahedron | 20 | 12 |
| **10** | 5 | 3 | 2 | truncated icosahedron = **standard soccer ball** | 12 | 20 |
| **11** | ≥ 3 | 2 | 2 | truncated beach ball | 2 | $k$ |
| **12** | 3 | 2 | 3 | variation on the tetrahedron | 4 | 6 |
| **13** | 4 | 2 | 3 | variation on the cube | 6 | 12 |
| **14** | 5 | 2 | 3 | variation on the dodecahedron | 12 | 30 |
| **15** | ≥ 3 | 1 | 3 | partially truncated beach ball | 1 | $k$ |
| **16** | ≥ 3 | 1 | 4 | double tin can | 2 | $2k$ |
| **17** | ≥ 3 | 1 | 5 | zigzag tin can | 2 | $2k$ |
| **18** | 3 | 1 | 6 | subdivision of the tetrahedron | 4 | 12 |
| **19** | 4 | 1 | 6 | subdivision of the cube | 6 | 24 |
| **20** | 5 | 1 | 6 | subdivision of the dodecahedron | 12 | 60 |


**Figure 9. Generalized soccer balls fall into 20 types. In this table, $k$ represents the number of sides in any black face; the product $m \times n$ is the number of sides in any white face. Every side of a black face meets a white face. Every $n$th side of a white face meets a black face. The columns $b$ and $w$ represent the number of black and white faces in the simplest representative of each type. For the types with $n$=2, every generalized soccer ball of that type is a branched covering of the simplest one. However, this is not true for other values of $n$. The minimal realization of type 8 is combinatorially the same as the 2006 World Cup ball shown in Figure 2, whereas type 10 is the standard soccer ball.**

other fullerenes: numbers 14 and 20, and the case $k$=6 of entry 17. The numbers of hexagons in these examples are 30, 60 and 2, respectively. (Note that in the latter case the color scheme is reversed, so the hexagons are black rather than white.) The numbers of carbon atoms are 80, 140 and 24, respectively. The last of these is the only fullerene with 24 atoms. In the case of 80 atoms, there are 7 different fullerenes with disjoint pentagons, but only one occurs in our table of generalized soccer balls. For 140 atoms, there are 121,354 fullerenes with disjoint pentagons.



**a** **b**

**Figure 10. The tetrahedron with just one black face *(a)* is the minimal realization of soccer-ball type 15 in Figure 9, where ($k$, $m$, $n$) = (3, 1, 3). Another realization, an octahedron with two opposite black faces *(b)*, is not a branched covering of *a*, showing that it is not possible to produce all generalized soccer balls with $n$>2 using branched coverings.**

Braungardt and I discovered something very intriguing when we tried to see whether every generalized soccer ball comes from a branched covering of one of the entries in our table. This is true, we found, for all the triples with $n$=2, that is, for generalized soccer balls for which black and white faces alternate around the sides of each white face. However, it is not true for other values of $n$! The easiest example demonstrating this failure arises for the triple ($k$, $m$, $n$) = (3, 1, 3), meaning that we have black and white triangles arranged in such a way that the sides of each black triangle meet only white ones, and each white triangle has exactly one side that meets a black one. The minimal example is just a tetrahedron with one face painted black *(Figure 10a)*. Another realization is an octahedron with two opposite faces painted black *(Figure 10b)*. This is not a branched covering of the painted tetrahedron! A branched covering of the tetrahedron would have 3, 6, 9, … faces meeting at every vertex—but the octahedron has 4.

The reason for this strange behavior is a subtle difference between the case $n$=2 and the cases $n$>2. In the tetrahedron example, there are two different kinds of vertices: a vertex at which only white faces meet, and three vertices where one black and two white faces meet. Moreover, the painted octahedron has yet another kind of vertex. But in the case $n$=2, all the vertices look essentially the same.


**Figure 11. Toroidal soccer balls are of two kinds: those that are branched coverings of spherical ones, and those that are not. A branched double cover of the standard spherical soccer ball produces a toroidal ball with 24 black and 40 white faces *(a)*. Opening up the standard soccer ball along two edges, deforming it to a tube and then matching the ends of the tube produces a toroidal soccer ball with 12 black and 20 white faces *(b)*. This pattern cannot be obtained as a branched covering.**


Every vertex has the same sequence of colors, which goes black, white, white, black, white, white, …, with only the length of the sequence left open. Thus the adjacency conditions provide a degree of control over the local structure of any generalized soccer ball with $n=2$. This control is lacking in the $n>2$ case. At present, therefore, it is possible to describe all generalized soccer balls with $n=2$: They are branched coverings of truncated Platonic solids. But there is no simple way to produce all the generalized soccer balls with $n>2$.

### Toroidal Soccer Balls

From a topologist's point of view, spherical soccer balls are just one particular example of maps drawn on surfaces. Because the definition of soccer balls through conditions (1), (2) and (3) does not specify that soccer-ball polyhedra should be spherical, there is a possibility that they might also exist in other shapes. Besides the sphere, there are infinitely many other surfaces that might occur: the torus (which is the surface of a doughnut), the double torus, the triple torus (which is the surface of a pretzel), the quadruple torus, etc. These surfaces are distinguished from one another by their *genus*, informally known as the number of holes: The sphere has genus zero, the torus has genus one, the double torus has genus two, and so on.

There are soccer balls of all genera, because every surface is a branched covering of the sphere (in a slightly more general way than we discussed before). By arranging the branch points to be vertices of some soccer ball graph on the sphere, we can generate soccer ball graphs on any surface. Figure 11a shows a toroidal soccer ball obtained from a two-fold branched covering of the standard spherical ball. In this case there are four branch points. Note that a two-fold branched covering always doubles the number of pentagons and hexagons.

Here is an easier construction of a toroidal soccer ball. Take the standard spherical soccer ball and cut it open along two disjoint edges. Opening up the sphere along each cut produces something that looks rather like a sphere from which two disks have been removed. This surface has a soccer-ball pattern on it, and the two boundary circles at which we have opened the sphere each have two vertices on them. If the cut edges are of the same type, meaning that along both of them two white faces met in the original spherical soccer ball, or that along both of them a black face met a white face, then we can glue the two boundary circles together so as to match vertices with vertices. (See Figure 11b for step-by-step illustrations of this construction.) The surface built in this way is again a torus. It has the structure of a polyhedron that satisfies conditions (1), (2) and (3), and is therefore a soccer ball.

This second toroidal soccer ball is not a branched covering of the standard spherical ball, because it has the same numbers of pentagons and hexagons (12 and 20 respectively) as the standard spherical ball. For a branched covering these numbers would be multiplied by the degree of the covering. In this case, the failure is not caused by loss of control over the local structure of the pattern (as in the previous section), but by a global property of the torus (the hole). Thus the basic result that all spherical soccer balls are branched coverings of the standard one is not true for soccer balls with holes.

### Coda

Soccer balls provide ample illustrations of the intimate connection that exists between graphs on surfaces and branched coverings. This circle of ideas is also connected to subtle questions in algebraic geometry, where the combinatorics of maps on surfaces encapsulates data from number theory in mysterious ways. Following the terminology introduced by Alexander Grothendieck, one of the leading mathematicians of the 20th century, the relevant graphs on the sphere are nowadays called *dessins d'enfants*.

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