SOLUTIONS TO HOMEWORK #2

1: $10100010 \longrightarrow 2^1 + 2^5 + 2^7 = 2 + 32 + 128 = \boxed{162}$

$00000000 \longrightarrow \boxed{0}$

$11001110 \longrightarrow 2^1 + 2^2 + 2^3 + 2^6 + 2^7 = 2 + 4 + 8 + 64 + 128$

$= 4 + 8 + 64 + 130$

$= 4 + 8 + 194$

$= 8 + 198 = \boxed{206}$

$00110100 \longrightarrow 2^2 + 2^4 + 2^5 = 4 + 16 + 32$

$= 20 + 32 = \boxed{52}$

$00100001 \longrightarrow 2^0 + 2^5 = 1 + 32 = \boxed{33}$

$17 = 16 + 1 = 2^4 + 2^0 \longrightarrow \boxed{10001}$

$25 = 16 + 8 + 1 = 2^4 + 2^3 + 2^0 \longrightarrow \boxed{11001}$

$31 = 16 + 8 + 4 + 2 + 1 = 2^4 + 2^3 + 2^2 + 2^1 + 2^0 \longrightarrow \boxed{11111}$

$40 = 32 + 8 = 2^5 + 2^3 \longrightarrow \boxed{101000}$

$63 = 32 + 31$

$= 32 + 16 + 8 + 4 + 2 + 1 = 2^5 + 2^3 + 2^2 + 2^1 + 2^0 \longrightarrow \boxed{111111}$

2: (NOTE THERES MORE THAN ONE CORRECT ANSWER)

$\{0, 1, 2, 3 \ldots\} \longrightarrow \{n-1 \mid n \in \mathbb{N}\}$

$\{\ldots, -3, 0, 3, 6 \ldots\} \longrightarrow \{3x \mid x \in \mathbb{Z}\}$

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\} \longrightarrow \{\frac{1}{n} \mid n \in \mathbb{N}\}$

$\{1, 2, 4, 8, 16, \ldots\} \longrightarrow \{2^{n-1} \mid n \in \mathbb{N}\}$

$\{7, 8, 9, \ldots\} \longrightarrow \{6+n \mid n \in \mathbb{N}\}$

$\{n^2 \mid n \in \mathbb{N}\} \longrightarrow \{1, 4, 9, 16 \ldots\}$

$\{5n+1 \mid n \in \mathbb{N}\} \longrightarrow \{6, 11, 16, 21, 26, 31 \ldots\}$

$\{x \in \mathbb{Z} \mid -2 < x\} \longrightarrow \{-1, 0, 1, 2 \ldots\}$

(CONT ON NEXT PAGE)

$$\{x \in \mathbb{Q} \mid x^2 = 9\} \longrightarrow \{3, -3\}$$

$$\{n^2 = 2 \mid n \in \mathbb{N}\} \longrightarrow \emptyset$$

3: $|\{1, 2, 3, 4, 5\}| = 5$

$|\{\ \}| = 0$

$|\{\text{RED}, \text{YELLOW}\} \cup \{\text{BLUE}\}| = 3$

$|\{\emptyset\}| = 1$

$|\{1, \{2, 3\}, \{1, 2, \{3\}\}\}| = 3$

$|\{\emptyset, \{\{1\}\}\}| = 2$

$|\emptyset \cap \{2, 3\}| = 0$

$|\{5, 1\} \cup \{1, 2, 5\}| = 3$

$|\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}\}| = 3$

PART 2

NOTE- $5^{-1} = \frac{1}{5} = \frac{2}{10} = 0.2$ BASE 10

$5^{-2} = \frac{1}{25} = \frac{4}{100} = 0.04$ BASE 10

$5^{-3} = \frac{1}{125} = \frac{4}{500} = \frac{8}{1000} = 0.008$ BASE 10

$5^{-4} = \frac{1}{625} = \frac{4}{2500} = \frac{8}{5000} = \frac{16}{10{,}000} = 0.0016$ BASE 10

SO $1.111 \longrightarrow 5^0 + 5^{-1} + 5^{-2} + 5^{-3} = 1 + 0.2 + 0.04 + 0.008$

$= \boxed{1.248}$

$1.234 \longrightarrow 5^0 + 2 \cdot 5^{-1} + 3 \cdot 5^{-2} + 4 \cdot 5^{-3} = 1 + 2(0.2) + 3(0.04) + 4(0.008)$

$= 1 + 0.4 + 0.12 + 0.032$

$= \boxed{1.552}$

$3.141 \longrightarrow 3 \cdot 5^0 + 1 \cdot 5^{-1} + 4 \cdot 5^{-2} + 1 \cdot 5^{-3}$

$= 3.0 + 0.2 + 4(0.04) + 1(0.008)$

$= 3.0 + 0.2 + 0.16 + 0.008$

$= \boxed{3.368}$

$$0.4321 \longrightarrow 4\cdot 5^{-1} + 3\cdot 5^{-2} + 2\cdot 5^{-3} + 1\cdot 5^{-4}$$

$$= 4(0.2) + 3(0.04) + 2(0.008) + 1(0.0016)$$

$$= 0.8 + 0.12 + 0.016 + 0.0016$$

$$= \boxed{0.9376}$$

b: $1.2 = 1 + 0.2 = 5^0 + 5^{-1} \longrightarrow \boxed{1.1}$

$4.4 = 4 + 2(0.2) = 4\cdot 5^0 + 2\cdot 5^{-1} \longrightarrow \boxed{4.2}$

$5.5 \circ 1 \longrightarrow \boxed{10.222\ldots\ldots}$

THE HARDER PART IS REPRESENTING ~~oa~~ $0.5 = \frac{1}{2}$
ONE GETS A REPEATING REPRESENTATION
LIKE THIS BECAUSE OF A SLICK ~~THE~~
FACTORING TRICK, BUT ANY SORT OF ARGUMENT
SHOWING $10.222\ldots 22$ IS TOO SMALL
AND $10.222\ldots 23$ IS TOO BIG IS FINE.

HERES THE TRICK: FOR ANY $k \in \mathbb{N}$

$$\left(1 - \frac{1}{5}\right)\left(\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots\ldots + \frac{1}{5^k}\right)$$

$$= \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots + \frac{1}{5^k}$$

$$- \frac{1}{25} - \frac{1}{125} - \quad - \frac{1}{5^k} - \frac{1}{5^{k+1}} = \frac{1}{5} - \frac{1}{5^{k+1}}$$

SO $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \dots + \frac{1}{5^k} = \dfrac{\frac{1}{5} - \frac{1}{5^{k+1}}}{1 - \frac{1}{5}}$

$$= \dfrac{\frac{5^k - 1}{5^{k+1}}}{\frac{4}{5}}$$

$$= \frac{5}{4} \, \frac{5^k - 1}{5^{k+1}} = \frac{1}{4} \, \frac{5^k - 1}{5^k}$$

$$= \frac{1}{4}\left(1 - \frac{1}{5^k}\right)$$

NOTE $\frac{5^k - 1}{5^k} < 1$.

THEN $\frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \dots + \frac{2}{5^k} = \frac{2}{4} \, \frac{5^k - 1}{5^k}$

$$= \frac{1}{2} \, \frac{5^k - 1}{5^k} < \frac{1}{2}$$

SO $0.222\dots22$ WILL ALWAYS REPRESENT A NUMBER LESS THAN A HALF, NO MATTER HOW LONG THE REPRESENTATION IS.

ON THE OTHER HAND:

$$\frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \dots + \frac{3}{5^k} = \frac{1}{2} \, \frac{5^k - 1}{5^k} + \frac{1}{5^k}$$

$$= \frac{1}{2} \, \frac{5^k - 1 + 2}{5^k}$$

$$= \frac{1}{2} \, \frac{5^k + 1}{5^k} > \frac{1}{2}$$

SO $0.222\dots2223$

ALWAYS REPRESENTS A NUMBER BIGGER THAN $\frac{1}{2}$. THE ONLY POSSIBILITY IS THAT $0.5 \longrightarrow 0.222\ldots$ (REPEATING) BASE 5.

THIS, IN FACT, WORKS: AS k GETS ENORMOUS

$\frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \cdots + \frac{2}{5^k} = \frac{1}{2}\,\frac{5^k - 1}{5^k}$ GETS CLOSER AND CLOSER TO $\frac{1}{2}$ SINCE $\frac{5^k-1}{5^k}$ IS GETTING CLOSER AND CLOSER TO 1.

FOR EXAMPLE: $\frac{5^{15}-1}{5^{15}} = \frac{30{,}517{,}578{,}124}{30{,}517{,}578{,}125}$

WHICH IS PRETTY CLOSE TO 1. THIS IS THE NOTION OF A LIMIT.

$3.14 \longrightarrow \boxed{3.03222\ldots \text{ (REPEATING)}}$ BASE 5

$3.12 \longrightarrow 3.03$ BASE 5. THE HARDER PART IS WRITING $0.02 = \frac{2}{100} = \frac{1}{50}$ BASE

BY THE SAME FACTORING TRICK:

$$\frac{2}{125} + \frac{2}{625} + \cdots + \frac{2}{5^k} = 2 \cdot \frac{\frac{1}{125} - \frac{1}{5^{k+1}}}{1 - \frac{1}{5}}$$

$$= 2 \cdot \frac{5}{4}\left(\frac{1}{125} - \frac{1}{5^{k+1}}\right)$$

$$= \frac{1}{2}\left(\frac{1}{25} - \frac{1}{5^k}\right)$$

AGAIN, AS k GETS BIG, $\frac{1}{5^k}$ WILL GET SMALL, SO $\frac{1}{2}\left(\frac{1}{25} - \frac{1}{5^k}\right)$ WILL GET CLOSE TO $\frac{1}{50} = 0.02$,

GIVING $3.14 \longrightarrow 3.03222\ldots$ BASE 5.

2a: YES. 5, 6, 7, 8, 9 ARE ALL EXAMLES OF NUMBERS WHOSE DOUBLE NEEDS TWICE AS MANY SYMBOLS.

2b: YES OR NO DEPENDING ON WHICH "DECIMAL SYSTEM" YOURE USING. IF YOU ONLY REPRESENT <u>NATURAL NUMBERS</u> IN YOUR SYSTEM, THEN DOUBLING CAN NEVER DECREASE THE NUMBER OF DIGITS; BECAUSE LARGER NUMBERS ALWAYS NEED AS MANY OR MORE DIGITS THAN SMALLER NUMBERS, AND DOUBLING MAKES A NUMBER BIGGER, THE NUMBER OF DIGITS CAN ONLY GO UP.

HOWEVER IF YOU ALLOW FRACTIONS (NEGATIVE POWERS LIKE IN THE LAST QUESTION), THEN 1.5 IS AN EXAMPLE.

3: A AND B ARE THE SAME.

GIVEN SOME $x \in A$ $x = 25a + 13b$ FOR SOME INTEGERS a AND b. THIS IS BY DEFINITION OF A. SINCE a IS AN INTEGER, SO IS $25a$. SIMILARLY ~~25b~~ $13b$ IS ALSO AN INTEGER. SINCE THE SUM OF TWO INTEGERS IS AN INTEGER

$x = 25a + 13b \in \mathbb{Z} = B$. SO A IS A SUBSET OF B.

GIVEN SOME $x \in B = \mathbb{Z}$, $-x \in \mathbb{Z}$ AND $2x \in \mathbb{Z}$ ALSO. SO $25(-x) + 13(2x) \in A$, BY DEFINITION OF A.

BUT $25(-x) + 13(2x) = x(-25 + 13(2))$

$$= x(-25 + 26)$$

$$= x.$$

SO $x \in A$, AND B IS A SUBSET OF A.

THEREFORE $A = B$.