ASSIGNMENT #3
SOLUTIONS

1: $\emptyset \subset A$ – ALWAYS TRUE

$A \subset \emptyset$ – SOMETIMES TRUE

$A \subset (A \cup B)$ – ALWAYS TRUE

$A \subset (A \cap B)$ – SOMETIMES TRUE

$A \cap (B \cap C) = (A \cap B) \cap C$ – ALWAYS TRUE

$A \cup (B \cup C) = (A \cup B) \cup C$ – ALWAYS TRUE

$A \cap \emptyset = A$ – SOMETIMES TRUE

$A \cup \emptyset = \emptyset$ – SOMETIMES TRUE

2: $100 = \frac{10}{1}$ RATIONAL

$\sqrt{49} = 7 = \frac{7}{1}$ RATIONAL

$10.2 =$ [redacted] $\frac{102}{10} = \frac{51}{5}$ RATIONAL

$10.213 = \frac{10213}{1000}$ RATIONAL

$10.2\overline{333} = 10.2 + 0.0\overline{333}$

$= \frac{102}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \ldots$

SUM GOES ON FOREVER

~~SEE~~ (SEE NEXT PAGE)

$$10.23\overline{33} = \frac{102}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$$

$$= \frac{102}{10} + \frac{3}{100}\left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots\right)$$

$$= \frac{102}{10} + \frac{3}{100}\sum_{j=0}^{\infty}\frac{1}{10^j}$$

$$= \frac{102}{10} + \frac{3}{100}\,\frac{1}{1-\frac{1}{10}}$$

$$= \frac{102}{10} + \frac{3}{100}\,\frac{1}{\frac{9}{10}}$$

$$= \frac{102}{10} + \frac{3}{100}\cdot\frac{10}{9} = \frac{102}{10} + \frac{1}{30}$$

$$= \frac{306+1}{30} = \boxed{\frac{307}{30}} \quad \text{(RATIONAL)}$$

(EASIER WAY THAT WORKS IN THIS CASE:

$$\frac{1}{3} = 0.33\overline{3} \implies \frac{1}{30} = 0.033\overline{3}$$

SO $10.23\overline{3} = \frac{102}{10} + \frac{1}{30} = \frac{307}{30}$)

$\sqrt{18} = 3\sqrt{2}$. IRRATIONAL

$$10.21\overline{21} = 10 + \frac{21}{100} + \frac{21}{1000} + \dots$$

$$= 10 + \frac{21}{100}\left(1 + \frac{1}{100} + \frac{1}{(100)^2} + \frac{1}{100^3} \dots\right.$$

$$= 10 + \frac{21}{100} \sum_{j=0}^{\infty} \frac{1}{(100)^j}$$

$$= 10 + \frac{21}{100} \frac{1}{1 - \frac{1}{100}}$$

$$= 10 + \frac{21}{100} \frac{1}{\frac{99}{100}}$$

$$= 10 + \frac{21}{100} \frac{100}{99}$$

$$= 10 + \frac{21}{99} = 10 + \frac{7}{33} = \boxed{\frac{337}{33}}$$ RATIONAL

$$5.214\overline{142} = 5.214 + \frac{1}{1000}\left(0.142\overline{142}\right)$$

$$= \frac{1}{1000}\left(5214 + 0.142\overline{142}\right)$$

$$= \frac{1}{1000}\left(5214 + \frac{142}{1000} + \frac{142}{(1000)^2} + \frac{142}{(1000)^3} + \dots\right)$$

$$= \frac{1}{1000}\left(5214 + \frac{142}{1000}\sum_{j=0}^{\infty}\frac{1}{(1000)^j}\right)$$

$$= \frac{1}{1000}\left(5214 + \frac{142}{1000}\,\frac{1}{1-\frac{1}{1000}}\right)$$

$$= \frac{1}{1000}\left(5214 + \frac{142}{1000}\,\frac{1000}{999}\right)$$

$$= \frac{1}{1000}\left(5214 + \frac{142}{999}\right)$$

$$= \frac{5214\cdot 999 + 142}{1000\cdot 999} = \boxed{\frac{651116}{124875}} \quad \text{RATIONAL}$$

COMPUTER ASSISTED

3: $\{\mathbb{N}, \mathbb{Q}, \mathbb{R}\}$ - ~~A~~ FINITE, CARDINALITY IS 3

$\{\emptyset\}$ - FINITE, CARDINALITY IS 1

$\{a^n \mid a, n \in \mathbb{Z}\}$ - COUNTABLY INFINITE (SUBSET OF $\mathbb{Q}$)

$\{(\mathbb{N} \cup \mathbb{Q}) \cup \mathbb{R}\} = \{\mathbb{R}\}$ - FINITE, CARDINALITY IS 1

HOWEVER: $(\mathbb{N} \cup \mathbb{Q}) \cup \mathbb{R} = \mathbb{R}$ WHICH IS UNCOUNTABLE

$\{\mathbb{R} \cap \mathbb{N}\} = \{\mathbb{N}\}$ - FINITE, CARDINALITY IS 1

HOWEVER $\mathbb{R} \cap \mathbb{N} = \mathbb{N}$ WHICH IS COUNTABLY INFINITE

$\{0 < x < 1 \mid x \in \mathbb{Q}\}$ - COUNTABLY INFINITE

$\{1 < x < 2 \mid x \in \mathbb{R}\}$ - UNCOUNTABLE

$\{\mathbb{P} \cap \{2n \mid n \in \mathbb{Z}\}\} = \{2\}$ - FINITE, CARDINALITY IS 1.

4: $n^3 - n = n(n^2 - 1) = n(n+1)(n-1)$

EITHER n OR $n+1$ IS EVEN, SO THE ~~PRODUCT~~ WHOLE EXPRESSION IS EVEN.

5: ASSUME $\sqrt{7}$ IS RATIONAL. THEN $\sqrt{7} = \frac{a}{b}$ WHERE a AND b HAVE NO COMMON FACTORS.

$\Longrightarrow 7b^2 = a^2$. IF b IS EVEN, THEN b^2 IS EVEN, AND SO $7b^2 = a^2$ IS EVEN, SO a IS EVEN, WHICH IS A CONTRADICTION, $\Longrightarrow$ b IS ODD

$\Longrightarrow b^2$ IS ODD $\Longrightarrow 7b^2$ IS ODD $\Longrightarrow a^2$ IS ODD

$\Longrightarrow a$ IS ODD.

WRITE ~~$a = 2d+1$~~ $a = 2c+1$ $\quad b = 2d+1$

THEN $7b^2 = 28d^2 + 28d + 7$

AND $a^2 = 4c^2 + 4c + 1$

SO $$28d^2 + 28d + 7 = 4c^2 + 4c + 1$$

$$\Longrightarrow 28d^2 + 28d + 6 = 4c^2 + 4c$$

$$\Longrightarrow 14d^2 + 14d + 3 = 2c^2 + 2c.$$

THIS CANT BE TRUE. THE LEFT SIDE IS ODD, BUT THE RIGHT SIDE IS EVEN. $\Longrightarrow$ b CANT BE EVEN OR ODD $\Longrightarrow \sqrt{7}$ IS IRRATIONAL.

6: LABEL THE SETS $A = \{\sqrt{x} \mid x \in \mathbb{Q}^+\}$

$B = \{\sqrt[3]{x} \mid x \in \mathbb{Q}^+\}$

NOTICE $A \cap B$ IS NOT EMPTY. (FOR EXAMPLE $\mathbb{N} \subset A \cap B$).

LET C BE THE SET OF ELEMENTS OF B THAT ARE NOT ELEMENTS OF A.

BY DEFINITION $C \subset B$. NOTICE C IS INFINITE: $\{\sqrt[3]{x} \mid x \text{ IS PRIME}\} \subset C$.

CONSIDER THE FOLLOWING 1-1 MATCHINGS BETWEEN $\mathbb{Q}^+$ AND A, B RESPECTIVELY.

$\alpha: \mathbb{Q}^+ \longrightarrow A$

VIA ~~$\alpha(x)$~~ $\alpha(x) = \sqrt{x}$

$\beta: \mathbb{Q}^+ \longrightarrow B$

VIA $\beta(x) = \sqrt[3]{x}$.

SINCE $\mathbb{Q}^+$ IS COUNTABLY INFINITE, SO ARE A AND B. C IS ALSO COUNTABLY INFINITE, SO THERE EXIST 1-1 MATCHINGS

$\gamma: \mathbb{N} \longrightarrow A$

AND $\delta: \mathbb{N} \longrightarrow C$.

THEN $\varepsilon : \mathbb{N} \longrightarrow A \cup B$ GIVEN BY

$$\varepsilon(n) = \begin{cases} \gamma\left(\frac{n}{2}\right) & n \text{ EVEN} \\ \delta\left(\frac{n+1}{2}\right) & n \text{ ODD} \end{cases}$$

IS A 1-1 MATCHING. $\Longrightarrow$ $A \cup B$ IS COUNTABLY INFINITE.

7: $A = \{2n \mid n \in \mathbb{N}\}$ (EVEN NUMBERS)

$B = \{2n-1 \mid n \in \mathbb{N}\}$ (ODD NUMBERS)

$C = \{-n \mid n \in \mathbb{N}\} \cup \{0\}$ (NON POSITIVE NUMBERS)

WILL DO. (THERE ARE MANY POSSIBLE ANSWERS)

8: THEY HAVE THE SAME CARDINALITY (UNCOUNTABLE)

$$x \longmapsto \frac{1-x}{x}$$

IS A 1-1 MATCHING BETWEEN $(0,1)$ AND $(0,\infty)$. (THERE ARE MANY OTHERS)