SOLUTIONS TO ASSIGNMENT 7

1a: TETRAHEDRON, 4 VERTICES, 6 EDGES, 4 FACES, $\deg v_i = 3$, $\deg f_i = 3$

b: CUBE, 8 VERTICES, 12 EDGES, 6 FACES $\deg v_i = 3$, $\deg f_i = 4$

c: OCTAHEDRON, 6 VERTICES, 12 EDGES, 8 FACES $\deg v_i = 4$, $\deg f_i = 3$

d: DODECAHEDRON, 20 VERTICES, 30 EDGES, 12 FACES, $\deg v_i = 3$, $\deg f_i = 5$

e: ICOSAHEDRON, 12 VERTICES, 30 EDGES 20 FACES $\deg v_i = 5$, $\deg f_i = 3$. 
2: This process describes taking the dual of the Platonic solids.

- The dual of the Tetrahedron is the Tetrahedron
- The dual of the Cube is the Octahedron
- The dual of the Octahedron is the Cube
- The dual of the Dodecahedron is the Icosahedron
- The dual of the Icosahedron is the Dodecahedron.

3: A three faces meeting at every corner means the degree of each vertex is three. We've already proved that the number of vertices of odd degree is even for any graph.
Since every vertex of the graph of a simple polyhedron has an odd degree (3), there must be an even number of vertices.

4: A graph with n vertices can have n vertices of degree zero. (Take a graph with no edges)

- A graph with n vertices can not have any vertices of degree n. For any vertex, there are only n-1 vertices to connect to.

- If a graph has a vertex of degree n-1, then there is an edge between that vertex and every other vertex. If every vertex is connected to at least one other vertex, there are no vertices of degree zero.
5: IN A GRAPH WITH $n$ VERTICES, IF EACH VERTEX WERE TO HAVE A DIFFERENT DEGREE, THERE WOULD BE ONE VERTEX OF DEGREE $n-1$, ONE OF DEGREE $n-2$, ONE OF DEGREE $n-3$, ..., ONE OF DEGREE 1, AND ONE OF DEGREE ZERO. THIS IS IMPOSSIBLE BY QUESTION 4.

6: FIRST NOTE THAT THE FORMULA HOLDS WHEN $n=1$: 

$$1(1+1)(2\cdot1+1) = \frac{2\cdot3}{6} = 1 = 1^2$$

Now assume the formula holds for some $n \in \mathbb{N}$, we need to show it also holds for $n+1$:

$$1^2 + 2^2 + 3^2 + \ldots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)(n(2n+1) + 6(n+1))}{6}$$

$$= \frac{n+1)(2n^2 + 7n + 6)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$