MODULI SPACES OF EQUIVARIANT h-COBORDISMS

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Classical parametrized stable h-cobordism theorem. Given a homotopy equivalence between smooth manifolds $M \simeq N$, the strategy to determine whether M and N are diffeomorphic is to try to construct an h-cobordism between them. By the classical h-cobordism (or s-cobordism) theorem, the obstructions to the h-cobordism being trivial, which would imply that $M \cong N$, are classified in terms of their Whitehead torsion $\tau \in Wh(\pi_1 M)$. The Whitehead group by definition is the quotient of $K_1(\pi_1 M)$ by $\pm g \in \pi_1 M$.

Theorem 0.1 ([2, 15, 16])**.** *Suppose M is a manifold with $\dim(M) \geq 5$. There is an isomorphism*

$$\{\textit{iso classes of } h\textit{-cobordisms on } M\} \cong Wh(\pi_1 M).$$

In order to study the space of all diffeomorphisms of M, it is necessary to topologize these obstructions [9, 18]. The Whitehead group $Wh(\pi_1 M)$ is the π_0 of the h-cobordism space $\mathcal{H}(M)$, whose k-simplices are h-cobordism bundles over Δ_k. The aforementioned theorem says that $\pi_0\mathcal{H}(M)$ can be computed in terms of K-theory. We can ask the same question about the higher homotopy groups of this moduli space:

Can we compute $\pi_i\mathcal{H}(M)$ in terms of algebraic K-theory?

The answer is yes, but only in a stable range, namely in the range where $\mathcal{H}(M)$ is equivalent to the stable version $\mathcal{H}^\infty(M)$ obtained by multiplying M by copies of I to increase its dimension [10]. This is the content of the celebrated "stable parametrized h-cobordism theorem."

Theorem 0.2 ([17])**.** *There is a decomposition*

$$\boldsymbol{A}(X) \simeq \Sigma^\infty X_+ \times \boldsymbol{Wh}(X), \tag{1}$$

where $\boldsymbol{Wh}(X)$ is a spectrum with the property that for a smooth compact manifold M, the underlying infinite loop space of $\Omega\,\boldsymbol{Wh}(M)$ is equivalent to the stable h-cobordism space $\mathcal{H}^\infty(M)$.

Weiss and Williams show that $\mathcal{H}^\infty(M)$ provides the information that accesses the diffeomorphism group of M in a stable range [18].

Equivariant h-cobordism spaces. Now suppose G is a finite group acting on a smooth manifold M with corners so that it has "trivial action on corners," namely M is a G-manifold modeled locally by $G \times_H V \times [0,\infty)^k$, for varying $H \leq G$ and H-representations V. Our goal is to understand the stable moduli space of equivariant h-cobordisms $\mathcal{H}^\infty_G(M)$ we constructed in [6] and show that it can be computed, at least in a range, by equivariant algebraic K-theory. Equivariant A-theory of a G-space $\mathbf{A}_G(X)$ was constructed in [13] using the machinery of spectral Mackey functors [7, 8, 5, 3, 4].[1] The main question/conjecture is the following.

Question 0.3. *For a compact smooth G-manifold M, is there a splitting*

$$\boldsymbol{A}_G(M) \simeq \Sigma^\infty_G M \times \boldsymbol{Wh}_G(M) \tag{2}$$

analogous to the nonequivariant one from equation (1), where $\Omega^{\infty+1}\,\boldsymbol{Wh}_G(M)^G \simeq \mathcal{H}^\infty_G(M)$?

An equivariant h-cobordism $(W; M, N)$ between compact G manifolds M and N is an h-cobordism W where the inclusions $M \hookrightarrow W$ and $N \hookrightarrow W$ are G-homotopy equivalences. For an equivariant parametrized stable h-cobordism theorem, we need to stabilize h-cobordisms with respect to representation disks. This is already apparent in the equivariant case on π_0: the equivariant Whitehead torsion of an equivariant h-cobordism $M \hookrightarrow W$ is the trivial element of the equivariant Whitehead group $Wh_G(M)$ if and only if there exists a G-representation V such that the equivariant h-cobordism $(W \times D(V); M \times D(V), N \times D(V))$ is trivial, where $D(V)$ is the unit disk in the representation V [1].

What underlies this result is the phenomenon that often equivariant results in manifold topology do not generalize unless there is a difference in the dimensions between fixed points. An equivariant map is isovariant if if preserves stabilizers, or fixed point strata. An equivariant h-cobordism is isovariant if the inclusion maps of the boundaries are isovariant homotopy equivalences. If a G-manifold satisfies the so-called "weak gap hypotheses" where the difference between dimensions of different fixed points is at least 3, then an equivariant h-cobordism on M is an isovariant h-cobordism [11]. The idea is that stabilizing with respect to representation disks increases the gaps between dimensions of fixed points.

For a compact smooth G-manifold M, denote by $M_{[H]}$ be compactification of the subspace of points with isotropy H, by removing tubular neighborhoods of smaller fixed-point submanifolds. Note that these will be manifolds with corners. Furthermore, denote by WH the Weyl group of H with respect to G. In work in progress, we prove the following unstable splitting result for isovariant h-cobordism spaces, which on π_0 recovers a result of Browder-Quinn and Rothenberg.

Theorem 0.4 (Goodwillie-Igusa-Malkiewich-M.)**.** *Let M be a compact smooth G-manifold. If $\dim M_{[H]}/WH \geq 5$, then the space of isovariant h-cobordisms satisfies a splitting*

$$\mathcal{H}^{\mathrm{iso}}_G(M) \simeq \prod_{(H)\leq G} \mathcal{H}(M_{[H]}/WH).$$

When we stabilize, the spaces of isovariant and equivariant h-cobordisms agree, and we obtain the following stable result about equivariant h-cobordism spaces.

Theorem 0.5 (Goodwillie-Igusa-Malkiewich-M.)**.** *Let M be a compact smooth G-manifold. Then the stable space of equivariant h-cobordisms (stabilized with respect*

[1] We note that $\mathbf{A}_G(X)$ is not the "K-theory of group actions for the G-action on the category of retractive spaces over X. That construction yields a G-spectrum that we call $\mathbf{A}^{\mathrm{coarse}}_G(X)$ and study further in [12].

to representation disks) satisfies a splitting

$$\mathcal{H}^\infty_G(M) \simeq \prod_{(H)\leq G} \mathcal{H}^\infty(M^H_{hWH}).$$

We can now ask the analogous question as before about this moduli space of equivariant h-cobordisms:

Can we describe $\mathcal{H}^\infty_G(M)$ in terms of equivariant algebraic K-theory?

Combined with the results of [14], we obtain a sequence in G-spectra

$$\mathbf{A}_G(M) \to \Sigma^\infty_G M \to \mathbf{Wh}_G(M)$$

where $\Omega^{\infty+1}\mathbf{Wh}_G(M)^G \simeq \mathcal{H}^\infty_G(M)$, a significant step toward Question 0.3.

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