VC 7.3

4. First, we evaluate $\int_S \nabla \times F \cdot dS$. Parameterize $S$ by

$$X(r, \theta) = \left( r \cos \theta, r \sin \theta, -\sqrt{4 - r^2} \right)$$

where $r$ goes from 0 to 2 and $\theta$ goes from 0 to $2\pi$. The tangent vectors are

$$T_r = \left( \cos \theta, \sin \theta, \frac{r}{\sqrt{4 - r^2}} \right)$$
$$T_\theta = (-r \sin \theta, r \cos \theta, 0)$$

and the normal vector is

$$T_\theta \times T_r = \begin{vmatrix} i & j & k \\ -r \sin \theta & r \cos \theta & 0 \\ \cos \theta & \sin \theta & \frac{r}{\sqrt{4 - r^2}} \end{vmatrix} = \frac{r^2}{\sqrt{4 - r^2}} \left( \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \right) - r \mathbf{k}.$$

Notice that, since $r \geq 0$, the $k$ component of this normal vector is $\leq 0$; we have the downward pointing normal.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y - z & x + y^2 - z & 4y - 3x \end{vmatrix} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

Now we can integrate:

$$\int_S \nabla \times F \cdot dS = \int_0^{2\pi} \int_0^2 \left( \frac{r^2}{\sqrt{4 - r^2}} \left( 5 \cos \theta + 2 \sin \theta \right) + r \right) dr \ d\theta$$
$$= \int_0^{2\pi} \left( 5 \cos \theta + 2 \sin \theta \right) d\theta \int_0^2 \frac{r^2}{\sqrt{4 - r^2}} dr + \int_0^{2\pi} \int_0^2 r \ dr \ d\theta$$
$$= (5 \sin \theta - 2 \cos \theta) \bigg|_0^{2\pi} \int_0^2 \frac{r^2}{\sqrt{4 - r^2}} dr + 2\pi \int_0^2 r \ dr$$
$$= 0 \int_0^2 \frac{r^2}{\sqrt{4 - r^2}} dr + 2\pi \left( \frac{r^2}{2} \right) \bigg|_0^2 = 4\pi.$$

Next, calculate $\oint_{\partial S} F \cdot ds$. The appropriate orientation for $\partial S$ is clockwise when viewed from the positive $k$ direction. A parameterization is

$$\mathbf{x}(t) = (2 \cos t, -2 \sin t, 0),$$

where $t$ goes from 0 to $2\pi$, and the tangent vector is

$$\mathbf{x}'(t) = (-2 \sin t, -2 \cos t, 0).$$
The line integral comes out to
\[ \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_{x} (2y - z) \, dx + (x + y^2 - z) \, dy + (4y - 3x) \, dz \]
\[ = \int_{0}^{2\pi} 2(-2 \sin t)(-2 \sin t) \, dt + (2 \cos t + 4 \sin^2 t)(-2 \cos t) \, dt \]
\[ = \int_{0}^{2\pi} (8 \sin^2 t - 4 \cos^2 t - 8 \sin^2 t \cos t) \, dt \]
\[ = \int_{0}^{2\pi} (4(1 - \cos 2t) - 2(1 + \cos 2t)) \, dt - 8 \int_{0}^{2\pi} \sin^2 t \cos t \, dt \]
\[ = \left(2 - 6 \cos 2t\right) \bigg|_{0}^{2\pi} - \frac{8}{3} \sin^3 t \bigg|_{0}^{2\pi} = 4\pi. \]

The content of Stokes’ theorem is that these two integrals are equal.

10. As suggested, consider a vector field \( \mathbf{F}(x, y, z) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} \), where \( M \) and \( N \) are scalar functions not depending on \( z \). Suppose that \( D \) is a closed, bounded region in the \( xy \)-plane with boundary \( C \) and orient \( C \) so that \( D \) is on the left as one traverses \( C \). This is the correct orientation to apply Stokes’ theorem if we choose the upward pointing normal for \( D \). By Stokes’ theorem,
\[ \iint_{D} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{C} \mathbf{F} \cdot d\mathbf{s} = \oint_{C} M \, dx + N \, dy. \] (1)

Calculate the left hand side.
\[ \nabla \times \mathbf{F} = \frac{\partial N}{\partial x} \mathbf{i} - \frac{\partial M}{\partial y} \mathbf{j} \bigg| = \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} \]

(Remember that \( M \) and \( N \) do not depend on \( z \), so \( \frac{\partial N}{\partial z} \) and \( \frac{\partial M}{\partial z} \) are 0.) Using \( x \) and \( y \) as parameters for \( D \), the tangent vectors are \( \mathbf{i} \) and \( \mathbf{j} \), respectively, so the normal vector is \( \mathbf{i} \times \mathbf{j} = \mathbf{k} \).

\[ \iint_{D} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_{D} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} \cdot \mathbf{k} \, dx \, dy \]
\[ = \iint_{D} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy \] (2)

Combining equations (1) and (2) we get Green’s theorem:
\[ \oint_{C} M \, dx + N \, dy = \iint_{D} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy. \]

16. Let \( D \) be the region in the plane \( 2x - 3y + 5z = 17 \) enclosed by the curve \( C \). Since the plane is a level set of the function \( f(x, y, z) = 2x - 3y + 5z \), we can get a normal vector by taking the gradient:
\[ \mathbf{N} = \nabla f = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}. \]
Notice that \( \mathbf{N} \) does not depend on \( x, y, \) or \( z \). We want to compute \( \int_C \mathbf{F} \cdot d\mathbf{s} \), where \( \mathbf{F} \) is the vector field

\[
(3 \cos x + z) \mathbf{i} + (5x - e^y) \mathbf{j} - 3y \mathbf{k}.
\]

Using Stokes’ theorem we get

\[
\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \nabla \times \mathbf{F} \cdot d\mathbf{S}.
\]

\[
\nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
3 \cos x + z & 5x - e^y & -3y
\end{vmatrix} = -3 \mathbf{i} + \mathbf{j} + 5 \mathbf{k}
\]

The unit normal vector to \( D \) is \( \pm \frac{\mathbf{N}}{||\mathbf{N}||} \); the sign is determined by the orientation of \( C \). Putting it all together,

\[
\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \nabla \times \mathbf{F} \cdot d\mathbf{S}
\]

\[
= \pm \frac{1}{||\mathbf{N}||} \iint_D (-3 \mathbf{i} + \mathbf{j} + 5 \mathbf{k}) \cdot (2 \mathbf{i} - 3 \mathbf{j} + 5 \mathbf{k}) \, dS
\]

\[
= \pm \frac{1}{||\mathbf{N}||} \iint_D 16 \, dS = \pm \frac{16}{||\mathbf{N}||} \text{(area of } D\text{)}.
\]

**DELA 2.1**

**T/F**

4. F

10. F

**Problems**

6. \( A \) is a 3 \( \times \) 3 matrix. The entries given are

\[
A = \begin{bmatrix}
-1 & 2 \\
3
\end{bmatrix}.
\]

Using \( a_{ji} = -a_{ij} \) we can get

\[
A = \begin{bmatrix}
-1 & 2 \\
1 & 3 \\
-2 & -3
\end{bmatrix}.
\]

The entries on the main diagonal have \( i = j \). For these elements, \( a_{ii} = -a_{ii} \), which means that \( a_{ii} = 0 \). Thus, the whole matrix is

\[
A = \begin{bmatrix}
0 & -1 & 2 \\
1 & 0 & 3 \\
-2 & -3 & 0
\end{bmatrix}.
\]
14. We assemble $B$ by writing $b_1$, $b_2$, $b_3$, and $b_4$ in its columns:

$$B = \begin{bmatrix} 2 & 5 & 0 & 1 \\ -1 & 7 & 0 & 2 \\ 4 & -6 & 0 & 3 \end{bmatrix}.$$ 

The row vectors of $B$ we get by reading horizontally:

$$[2 \ 5 \ 0 \ 1], \quad [-1 \ 7 \ 0 \ 2], \quad \text{and} \quad [4 \ -6 \ 0 \ 3].$$

20. In order to be lower triangular, our matrix $A$ needs zeros above the main diagonal.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

To make $A$ skew-symmetric, the entries below the main diagonal should be the negatives of those above:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Finally, a skew-symmetric matrix needs zeros on the main diagonal, since $a_{ii} = -a_{ii}$.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So in fact this matrix of zeros is the only $3 \times 3$ lower triangular skew-symmetric matrix.

22. A good example of a function that hits the same value twice but not three times is a quadratic function. If $f(t) = t(t - 1)$ then $f(0) = f(1) \neq f(2)$, since $f(0)$ and $f(1)$ are 0 and quadratic functions have no more than 2 roots. Any constant multiple of $f$ will have this same property. So for $A$ we could pick

$$A = \begin{bmatrix} t(t - 1) & 2t(t - 1) & 3t(t - 1) \\ 4t(t - 1) & 5t(t - 1) & 6t(t - 1) \\ 7t(t - 1) & 8t(t - 1) & 9t(t - 1) \end{bmatrix}.$$ 

**DELA 2.2**

T/F

4. F
Problems

2. If \(2A + B - 3C + 2D = A + 4C\), then we can rearrange to get

\[
D = \frac{1}{2}(-A - B + 7C).
\]

Now, substitute the given matrices for \(A\), \(B\), and \(C\) and simplify:

\[
D = \frac{1}{2}(-A - B + 7C) \\
= \frac{1}{2} \left( \begin{bmatrix} -2 & 1 & 0 \\ 3 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} + 7 \begin{bmatrix} -1 & -1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{bmatrix} \right) \\
= \frac{1}{2} \left( \begin{bmatrix} -10 & -5 & 5 \\ -3 & -1 & -2 \\ 1 & -1 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & -2 \\ -3 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} -7 & -7 & 7 \\ 7 & 14 & 21 \\ -7 & 7 & 0 \end{bmatrix} \right) \\
= \frac{1}{2} \begin{bmatrix} -5 & -\frac{5}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{13}{2} & 9 \\ -\frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}.
\]

10. The vector \(Ac\) is a linear combination of the column vectors of \(A\). The coefficients of this linear combination are the entries of \(c\). The column vectors of \(A\) are

\[
\begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ -6 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}.
\]

Now we can compute

\[
Ac = 2 \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ -6 \end{bmatrix} - 4 \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \\
= \begin{bmatrix} 6 \\ 4 \\ 14 \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \\ -18 \end{bmatrix} + \begin{bmatrix} -16 \\ -20 \\ -12 \end{bmatrix} \\
= \begin{bmatrix} -13 \\ -13 \\ -16 \end{bmatrix}.
\]

24. If \(A\) and \(C\) are \(m \times n\) matrices, we aim to prove that

\[
(A^T)^T = A
\]  \hspace{1cm} (1)

and

\[
(A + C)^T = A^T + C^T.
\]  \hspace{1cm} (2)

Let \(a_{ij}\) be the entries of the matrix \(A\), with \(1 \leq i \leq m\) and \(1 \leq j \leq n\), and let \(c_{ij}\) be the entries of \(C\).
To prove (1), let \( B = A^T \). Since \( A \) is \( m \times n \), it follows that \( B \) is \( n \times m \). If \( b_{ij} \) are the entries of \( B \), with \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), then \( b_{ij} = a_{ji} \). Now, let \( D = B^T \), so that \( D = (A^T)^T \). The matrix \( B \) is \( n \times m \), so \( D \) will be \( m \times n \). The entries of \( D \) are \( d_{ij} = b_{ji} \), with \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). The dimensions of \( D \) and \( A \) are the same—both are \( m \times n \) matrices. Furthermore, \( d_{ij} = b_{ji} = a_{ij} \), so we can conclude that \( D = A \), that is, we have proved (1).

Let’s move on to (2). If \( E = A + C \) and \( e_{ij} \) are its entries then \( e_{ij} = a_{ij} + c_{ij} \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). Its transpose is the \( n \times m \) matrix with entries \( e_{ji} \). On the other hand, the entries of \( A^T \) and \( C^T \) are \( a_{ji} \) and \( c_{ji} \), respectively, so the entries of \( F = A^T + C^T \) are \( f_{ij} = a_{ji} + c_{ji} \). Since \( A \) and \( C \) are \( m \times n \) matrices, \( A^T, C^T \) and \( F \) are \( n \times m \). For a start the dimensions of \( E^T \) agree with those of \( F \). As for the entries, \( f_{ij} = a_{ji} + c_{ji} = e_{ji} \), so we conclude that \( F = E^T \), which proves (2).

36. (a) In order for \( AA^T \) to be symmetric we need \((AA^T)^T = AA^T\). Well, part (3) of Theorem 2.2.21 tells us that

\[
(AA^T)^T = (A^T)^T A^T
\]

and then we can use part (1) to get

\[
(A^T)^T A^T = AA^T.
\]

(b) To show \((ABC)^T = C^TB^TC^T\), use part (3) of Theorem 2.2.21 twice:

\[
(ABC)^T = C^T(AB)^T = C^TB^TA^T.
\]

38. To differentiate a matrix function, take the derivative of each entry:

\[
\frac{d}{dt} t = 1,
\]

\[
\frac{d}{dt} \sin t = \cos t,
\]

\[
\frac{d}{dt} \cos t = -\sin t, \quad \text{and}
\]

\[
\frac{d}{dt} 4t = 4.
\]

Thus,

\[
\frac{dA}{dt} = \left[ \begin{array}{cc} 1 & \cos t \\ -\sin t & 4 \end{array} \right].
\]

42. We integrate each entry of the matrix function:

\[
\int_0^\frac{\pi}{2} \cos t \, dt = \sin t \bigg|_0^{\frac{\pi}{2}} = 1 \quad \text{and}
\]

\[
\int_0^\frac{\pi}{2} \sin t \, dt = -\cos t \bigg|_0^{\frac{\pi}{2}} = 1,
\]

so the integral of \( A \) is

\[
\int_0^\frac{\pi}{2} A(t) \, dt = \left[ \begin{array}{c} 1 \\ 1 \end{array} \right].
\]