Math 114 Review

Math 240 — Calculus III

Summer 2013, Session II

Monday, July 1, 2013
1. Gradient, Divergence, and Curl
   Gradient
   Divergence
   Curl
   How they’re related

2. Line integrals
   Scalar line integrals
   Vector line integrals
   Conservative vector fields
Definition

Let $f : X \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$ be a differentiable scalar function on a region of 3-dimensional space. The \textbf{gradient} of $f$ is the vector field

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}.$$ 

The direction of the gradient, $\frac{\nabla f}{\|\nabla f\|}$, is the direction in which $f$ is increasing the fastest. The norm, $\|\nabla f\|$, is the rate of this increase.

Example

If $f(x, y, z) = x^2 + y^2 + z^2$ then

$$\nabla f = 2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k}.$$
**Definition**

Let \( F : X \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be a differentiable vector field with components \( F = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} \). The **divergence** of \( F \) is the scalar function

\[
\text{div } F = \nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.
\]

The divergence of a vector field measures how much it is “expanding” at each point.

**Examples**

1. If \( F = x \mathbf{i} + y \mathbf{j} \) then \( \nabla \cdot F = 2 \).
2. If \( F = -y \mathbf{i} + x \mathbf{j} \) then \( \nabla \cdot F = 0 \).
Definition
Let \( \mathbf{F} : X \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be a differentiable vector field with components \( \mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} \). The \textbf{curl} of \( \mathbf{F} \) is the vector field

\[
\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_x & F_y & F_z
\end{vmatrix}
\]

\[
= \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathbf{i} + \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathbf{j} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{k}.
\]

The magnitude of the curl, \( \| \nabla \times \mathbf{F} \| \), measures how much \( \mathbf{F} \) rotates around a point. The direction of the curl, \( \frac{\nabla \times \mathbf{F}}{\| \nabla \times \mathbf{F} \|} \), is the axis around which it rotates.
Example

If $\mathbf{F} = -y \, \mathbf{i} + x \, \mathbf{j}$ then $\nabla \times \mathbf{F} = 2 \, \mathbf{k}.$
Let $f : X \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a $C^2$ scalar function. Then
\[ \nabla \times (\nabla f) = 0, \text{ that is, } \text{curl (grad } f) = 0. \]

Let $F : X \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a $C^2$ vector field. Then
\[ \nabla \cdot (\nabla \times F) = 0, \text{ that is, } \text{div (curl } F) = 0. \]

To summarize, the composition of any two consecutive arrows in the diagram yields zero.
Definition
Let \( \mathbf{x} : [a, b] \rightarrow X \subseteq \mathbb{R}^3 \) be a \( C^1 \) path and \( f : X \rightarrow \mathbb{R}^3 \) a continuous function. The **scalar line integral** of \( f \) along \( \mathbf{x} \) is

\[
\int_{\mathbf{x}} f \, ds = \int_{a}^{b} f(\mathbf{x}(t)) \left\| \mathbf{x}'(t) \right\| \, dt.
\]

In two dimensions, a scalar line integral measures the area under a curve with base \( \mathbf{x} \) and height given by \( f \).
Example

Let $\mathbf{x} : [0, 2\pi] \to \mathbb{R}^3$ be the helix $\mathbf{x}(t) = (\cos t, \sin t, t)$ and let $f(x, y, z) = xy + z$. Let’s compute

$$\int_{\mathbf{x}} f \, ds = \int_0^{2\pi} f(\mathbf{x}(t)) \|\mathbf{x}'(t)\| \, dt.$$ 

We find

$$\|\mathbf{x}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2},$$

so now

$$\int_0^{2\pi} f(\mathbf{x}(t)) \|\mathbf{x}'(t)\| \, dt = \int_0^{2\pi} (\cos t \sin t + t) \sqrt{2} \, dt$$

$$= \sqrt{2} \int_0^{2\pi} (\frac{1}{2} \sin 2t + t) \, dt = 2\sqrt{2}\pi^2.$$
**Definition**

Let \( x : [a, b] \rightarrow X \subseteq \mathbb{R}^3 \) be a \( C^1 \) path and \( F : X \rightarrow \mathbb{R}^3 \) a continuous vector field. The **vector line integral** of \( F \) along \( x \) is

\[
\int_{x} F \cdot ds = \int_{a}^{b} F(x(t)) \cdot x'(t) \, dt.
\]

If \( F \) has components \( F = F_x i + F_y j + F_z k \), the vector line integral can also be written

\[
\int_{x} F \cdot ds = \int_{x} F_x dx + F_y dy + F_z dz.
\]

Physically, a vector line integral measures the work done by the force field \( F \) on a particle moving along the path \( x \).
Example

Let \( \mathbf{x} : [0, 1] \to \mathbb{R}^3 \) be the path \( \mathbf{x}(t) = (2t + 1, t, 3t - 1) \) and let \( \mathbf{F} = -z \mathbf{i} + x \mathbf{j} + y \mathbf{k} \). Let's compute

\[
\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} -z \, dx + x \, dy + y \, dz.
\]

First, we find \( \mathbf{x}'(t) = (2, 1, 3) \), and now we can do

\[
\int_{\mathbf{x}} -z \, dx + x \, dy + y \, dz = \int_{0}^{1} -(3t - 1)(2) + (2t + 1) + t(3) \, dt
\]

\[
= \int_{0}^{1} -t + 3 \, dt = \frac{5}{2}.
\]
Example 7: If \( x: \mathbb{R} \to \mathbb{R}^2 \) is any path and let \( y: \mathbb{R} \to \mathbb{R}^2 \) be a continuous function whose domain contains the image of \( x \). If \( t \) is any reparametrization of \( x \) and \( u \) is any reparametrization of \( y \), then

\[
\int_y f \, ds = \int_x f \, ds.
\]

\[
\int_y F \cdot ds = -\int_x F \cdot ds.
\]

This can be achieved by negating \( t \):

\[
y(t) = x(-t).
\]
Definition
A continuous vector field $\mathbf{F}$ is called a **conservative vector field**, or a **gradient field**, if $\mathbf{F} = \nabla f$ for some $C^1$ scalar function $f$. In this case we also say that $f$ is a **scalar potential** of $\mathbf{F}$.

Theorem
*Suppose $\mathbf{F}$ is a continuous vector field defined on a connected, open region $R \subseteq \mathbb{R}^3$. Then $\mathbf{F} = \nabla f$ if and only if $\mathbf{F}$ has path independent line integrals in $R$.***
We say \( \mathbf{F} : R \subseteq \mathbb{R}^3 \to \mathbb{R}^3 \) has **path independent line integrals** if any of the following hold:

1. \( \int_x \mathbf{F} \cdot ds = \int_y \mathbf{F} \cdot ds \) whenever \( x \) and \( y \) are two simple \( C^1 \) paths in \( R \) with the same initial and terminal points,

2. \( \oint_x \mathbf{F} \cdot ds = 0 \) for any simple, closed \( C^1 \) path \( x \) lying in \( R \) (meaning the initial and terminal points of \( x \) coincide),

3. \( \int_C \mathbf{F} \cdot ds = f(B) - f(A) \) for any differentiable curve \( C \) in \( R \) running from point \( A \) to point \( B \), and for any scalar potential \( f \).
To justify our terminology, if $f$ is a scalar potential for the vector field $\mathbf{F}$, it means that we can interpret $f$ as measuring the potential energy associated with the force represented by $\mathbf{F}$.

In this setting, criterion 3 from the previous slide says that
\[
\text{work} = \int_C \mathbf{F} \cdot d\mathbf{s} = f(B) - f(A) = \text{change in potential energy},
\]
meaning that the force represented by $\mathbf{F}$ obeys conservation of energy.
A test for conservative fields

**Theorem**

Suppose \( \mathbf{F} \) is a \( C^1 \) vector field defined in a **simply-connected** region, \( R \), (intuitively, \( R \) has no holes going all the way through). Then \( \mathbf{F} = \nabla f \) for some \( C^2 \) scalar function if and only if \( \nabla \times \mathbf{F} = 0 \) at all points in \( R \).

**Example**

Let

\[
\mathbf{F} = \left( \frac{x}{x^2+y^2+z^2} - 6x \right) \mathbf{i} + \frac{y}{x^2+y^2+z^2} \mathbf{j} + \frac{z}{x^2+y^2+z^2} \mathbf{k}.
\]

\( \mathbf{F} \) is \( C^1 \) on \( \mathbb{R}^3 - \{(0,0,0)\} \), which is a simply-connected domain. Check that

\[
\nabla \times \mathbf{F} = 0
\]

everywhere \( \mathbf{F} \) is defined. Therefore, \( \mathbf{F} \) is conservative.