Surface Integrals

Math 240 — Calculus III

Summer 2013, Session II

Wednesday, July 3, 2013
1. Scalar surface integrals
   Surface area

2. Vector surface integrals

3. Changing orientation
Definition

Let $X : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a smooth parameterized surface. Let $f$ be a continuous scalar function whose domain includes $S = X(D)$. The **scalar surface integral** of $f$ along $X$ is

$$
\int \int_X f \, dS = \int \int_D f(X(s, t)) \|T_s \times T_t\| \, ds \, dt
$$

$$
= \int \int_D f(X(s, t)) \|N(s, t)\| \, ds \, dt.
$$
Example

Let $S$ be the closed cylinder of radius 3 with axis along the $z$-axis, top face at $z = 15$, and bottom face at $z = 0$. Let’s calculate $\iint_S z \, dS$. Denote the lateral cylindrical face of $S$ by $S_1$ and the bottom and top faces by $S_2$ and $S_3$, respectively.

We compute

$$\iint_{S_1} z \, dS = 675\pi,$$
$$\iint_{S_2} z \, dS = 0,$$
$$\iint_{S_3} z \, dS = 135\pi.$$ 

Therefore,

$$\iint_S z \, dS = \iint_{S_1} z \, dS + \iint_{S_2} z \, dS + \iint_{S_3} z \, dS = 810\pi.$$
**Fact**

If $S$ is a smooth surface parameterized by $\mathbf{X} : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$ then the surface area of $S$ is given by

$$
\iint_{D} \| \mathbf{N} \| \, ds \, dt = \iint_{D} \| \mathbf{T}_s \times \mathbf{T}_t \| \, ds \, dt = \iint_{\mathbf{X}} 1 \, dS.
$$

*Figure:* The quantity $\| \mathbf{T}_s \times \mathbf{T}_t \|$ is the area of the gray square on the right.
Example

Recall our parameterization of a sphere:

\[ X(s, t) = r (\cos s)(\sin t) \mathbf{i} + r (\sin s)(\sin t) \mathbf{j} + r (\cos t) \mathbf{k}. \]

We calculate

\[ T_s = -r \sin s \sin t \mathbf{i} + r \cos s \sin t \mathbf{j}, \]
\[ T_t = r \cos s \cos t \mathbf{i} + r \sin s \cos t \mathbf{j} - r \sin t \mathbf{k}, \]
\[ N = -r^2 \cos s \sin^2 t \mathbf{i} - r^2 \sin s \sin^2 t \mathbf{j} - r^2 \sin t \cos t \mathbf{k}, \]
and \[ ||N|| = r^2 \sin t. \]

Therefore, the surface area of the sphere is

\[ \int_0^\pi \int_0^{2\pi} r^2 \sin t \, ds \, dt = \int_0^\pi 2\pi r^2 \sin t \, dt = 4\pi r^2. \]
Definition
Let \( X : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3 \) be a smooth parameterized surface. Let \( F \) be a continuous vector field whose domain includes \( S = X(D) \). The **vector surface integral** of \( F \) along \( X \) is

\[
\iint_X F \cdot dS = \iint_D F(X(s, t)) \cdot N(s, t) \, ds \, dt.
\]

In physical terms, we can interpret \( F \) as the flow of some kind of fluid. Then the vector surface integral measures the volume of fluid that flows through \( S \) per unit time. This is called the **flux** of \( F \) across \( S \).
The parametrized surface $Y$ is the same as $X$, except that the standard normal vector arising from $Y$ points in the opposite direction to the one arising from $X$.

The calculation in Example 7 generalizes thus: Suppose $X$ is a smooth parametrized surface and $Y$ is a smooth reparametrization of $X$ via $H$, meaning that $Y(s,t) = X(u,v) = X(H(s,t))$.

Since $H$ is assumed to be of class $C^1$, we can show from the chain rule that the standard normal vectors are related by the equation

$$N_Y(s,t) = \frac{\partial(u,v)}{\partial(s,t)} N_X(u,v).$$

(11)

(See the addendum at the end of this section for a derivation of formula (11).)

Formula (11) shows that $N_Y$ is a scalar multiple of $N_X$. In addition, since $H$ is invertible and both $H$ and $H^{-1}$ are of class $C^1$, it follows that the Jacobian of $H$ is either always positive or always negative. (To see this, note that both $H \circ H^{-1}$ and $H^{-1} \circ H$ are the identity function. Hence, the chain rule may be applied to show that the derivative matrix $D_H(s,t)$ is invertible for each $(s,t)$; therefore, its determinant, which is the Jacobian of $H$, must be nonzero. Since the determinant is a continuous function of the entries of $H$, it thus cannot change sign.) Hence, the standard normal $N_Y$ either always points in the same direction as $N_X$ or else always points in the opposite direction (Figure 7.20).

Under these assumptions, we say that both $H$ and $Y$ are orientation-preserving if the Jacobian $\frac{\partial(u,v)}{\partial(s,t)}$ is positive, orientation-reversing if $\frac{\partial(u,v)}{\partial(s,t)}$ is negative.

The following result, a close analogue of Theorem 1.4, Chapter 6, shows that smooth reparametrization has no effect on the value of a scalar line integral.

**Theorem 2.4**

Let $X: D_1 \to \mathbb{R}^3$ be a smooth parametrized surface and $f$ any continuous function whose domain includes $X(D_1)$. If $Y: D_2 \to \mathbb{R}^3$ is any smooth reparametrization of $X$, then

$$\int \int_Y f \, dS = \int \int_X f \, dS.$$

This can be achieved by exchanging $s$ and $t$:

$$T_t \times T_s = - (T_s \times T_t).$$

Figure: $X$ and $Y$ parameterize the same surface with opposite normal directions.