Stokes’ and Gauss’ Theorems

Math 240 — Calculus III

Summer 2013, Session II

Monday, July 8, 2013
1. Stokes’ theorem

2. Gauss’ theorem
   Calculating volume with Gauss’ theorem
Theorem (Green’s theorem)

Let $D$ be a closed, bounded region in $\mathbb{R}^2$ with boundary $C = \partial D$. If $\mathbf{F} = M \mathbf{i} + N \mathbf{j}$ is a $C^1$ vector field on $D$ then

$$\oint_C M \, dx + N \, dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy.$$  

Notice that $\left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} = \nabla \times \mathbf{F}$.

Theorem (Stokes’ theorem)

Let $S$ be a smooth, bounded, oriented surface in $\mathbb{R}^3$ and suppose that $\partial S$ consists of finitely many $C^1$ simple, closed curves. If $\mathbf{F}$ is a $C^1$ vector field whose domain includes $S$, then

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$
Stokes’ theorem and orientation

**Definition**
A smooth, connected surface, $S$ is **orientable** if a nonzero normal vector can be chosen continuously at each point.

**Examples**
- **Orientable** planes, spheres, cylinders, most familiar surfaces
- **Nonorientable** Möbius band

To apply Stokes’ theorem, $\partial S$ must be correctly oriented.

Right hand rule: thumb points in chosen normal direction, fingers curl in direction of orientation of $\partial S$.

Alternatively, when looking down from the normal direction, $\partial S$ should be oriented so that $S$ is on the *left*.
Stokes’ theorem

Example

Let $S$ be the paraboloid $z = 9 - x^2 - y^2$ defined over the disk in the $xy$-plane with radius 3 (i.e. for $z \geq 0$). Verify Stokes’ theorem for the vector field

$$\mathbf{F} = (2z - y) \mathbf{i} + (x + z) \mathbf{j} + (3x - 2y) \mathbf{k}.$$  

We calculate

$$\nabla \times \mathbf{F} = -3 \mathbf{i} - \mathbf{j} + 2 \mathbf{k} \quad \text{and} \quad \mathbf{N} = 2x \mathbf{i} + 2y \mathbf{j} + \mathbf{k}.$$  

Therefore,

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D (-6x - 2y + 2) \, dx \, dy = 18\pi.$$  

Example

Let $S$ be the paraboloid $z = 9 - x^2 - y^2$ defined over the disk in the $xy$-plane with radius 3 (i.e. for $z \geq 0$). Verify Stokes' theorem for the vector field

$$\mathbf{F} = (2z - y) \mathbf{i} + (x + z) \mathbf{j} + (3x - 2y) \mathbf{k}.$$ 

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D (-6x - 2y + 2) \, dx \, dy = 18\pi.$$ 

Using Stokes' theorem, we can do instead

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \oint_C -y \, dx + x \, dy$$

$$= \int_0^{2\pi} (-3 \sin t)^2 + (3 \cos t)^2 \, dt = 18\pi.$$
Example

Let \( S \) be the paraboloid \( z = 9 - x^2 - y^2 \) defined over the disk in the \( xy \)-plane with radius 3 (i.e. for \( z \geq 0 \)). Verify Stokes’ theorem for the vector field \( \mathbf{F} = (2z - y) \mathbf{i} + (x + z) \mathbf{j} + (3x - 2y) \mathbf{k} \).

\[
\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D (-6x - 2y + 2) \, dx \, dy = 18\pi.
\]

Applying Stokes’ theorem a second time yields

\[
\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \nabla \times \mathbf{F} \cdot d\mathbf{S}
\]
\[
= \iint_D 2 \, d\mathbf{S} = 2 \text{ (area of } D) = 18\pi.
\]
Theorem (Gauss’ theorem, divergence theorem)

Let $D$ be a solid region in $\mathbb{R}^3$ whose boundary $\partial D$ consists of finitely many smooth, closed, orientable surfaces. Orient these surfaces with the normal pointing away from $D$. If $\mathbf{F}$ is a $C^1$ vector field whose domain includes $D$ then

$$
\iint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F} \, dV.
$$
Example

Let $\mathbf{F}$ be the radial vector field $x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$ and let $D$ be the solid cylinder of radius $a$ and height $b$ with axis on the $z$-axis and faces at $z = 0$ and $z = b$. Let's verify Gauss' theorem. Let $S_1$ and $S_2$ be the bottom and top faces, respectively, and let $S_3$ be the lateral face.

To orient $\partial D$ for Gauss' theorem, choose normals

$$\mathbf{n}_1 = -\mathbf{k} \text{ for } S_1, \quad \mathbf{n}_2 = \mathbf{k} \text{ for } S_2, \quad \text{and } \mathbf{n}_3 = \frac{1}{a}(x \mathbf{i} + y \mathbf{j}) \text{ for } S_3.$$

Now we integrate over the surface

$$\iiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = b \iint_{S_2} dS + a \iiint_{S_3} dS = 3\pi a^2 b.$$
Example

Let \( \mathbf{F} \) be the radial vector field \( x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \) and let \( D \) the be solid cylinder of radius \( a \) and height \( b \) with axis on the \( z \)-axis and faces at \( z = 0 \) and \( z = b \). Let’s verify Gauss’ theorem. Let \( S_1 \) and \( S_2 \) be the bottom and top faces, respectively, and let \( S_3 \) be the lateral face.

\[
\iiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = b \iint_{S_2} dS + a \iint_{S_3} dS = 3\pi a^2 b.
\]

On the other hand, \( \nabla \cdot \mathbf{F} = 3 \).

Then

\[
\iiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_{D} \nabla \cdot \mathbf{F} \, dV = 3 \iiint_{D} dV = 3\pi a^2 b.
\]
Recall how we used Green’s theorem to calculate the area of a plane region via a line integral around its boundary.

**Theorem**

Suppose $D$ is a solid region in $\mathbb{R}^3$ to which Gauss’ theorem applies and $\mathbf{F}$ is a $C^1$ vector field such that $\nabla \cdot \mathbf{F}$ is identically 1 on $D$. Then the volume of $D$ is given by

$$\iiint_{\partial D} \mathbf{F} \cdot d\mathbf{S}$$

where $\partial D$ is oriented as in Gauss’ theorem.

Some examples are

$$\text{Volume of } D = \begin{cases} \iiint_{\partial D} (x \mathbf{i}) \cdot d\mathbf{S} \\ \iiint_{\partial D} (y \mathbf{j}) \cdot d\mathbf{S} \\ \iiint_{\partial D} (z \mathbf{k}) \cdot d\mathbf{S} \end{cases}$$
Example

Let’s calculate the volume of a truncated cone via an integral over its surface. Let $D$ be the solid bounded by the cone

$$x^2 + y^2 = (2 - z)^2$$

and the planes $z = 1$ and $z = 0$. Let’s use the vector field $\mathbf{F} = x \mathbf{i}$, so that $\int \int_S \mathbf{F} \cdot d\mathbf{S} = 0$ when $S$ is the top or bottom face. Then we just need to calculate

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -1 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = x \mathbf{i} + y \mathbf{j} + r \mathbf{k}$$

and the volume of $D$ is

$$\int \int_D (x \mathbf{i}) \cdot d\mathbf{S} = \int_0^{2\pi} \int_1^2 (r \cos \theta)^2 \, dr \, d\theta = \frac{7}{3} \pi.$$