1. Eigenvalues and Eigenvectors

2. Diagonalization
Next week, we will apply linear algebra to solving differential equations. One that is particularly easy to solve is

\[ y' = ay. \]

It has the solution \( y = ce^{at} \), where \( c \) is any real (or complex) number. Viewed in terms of linear transformations, \( y = ce^{at} \) is the solution to the vector equation

\[ T(y) = ay, \quad (1) \]

where \( T : C^k(I) \rightarrow C^{k-1}(I) \) is \( T(y) = y' \). We are going to study equation (1) in a more general context.
**Definition**

Let $A$ be an $n \times n$ matrix. Any value of $\lambda$ for which

$$Av = \lambda v$$

has *nontrivial* solutions $v$ are called **eigenvalues** of $A$. The corresponding *nonzero* vectors $v$ are called **eigenvectors** of $A$.

![Figure: A geometrical description of eigenvectors in $\mathbb{R}^2$.](image-url)
Example

If $A$ is the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -3 & 5 \end{bmatrix},$$

then the vector $v = (1, 3)$ is an eigenvector for $A$ because

$$A v = \begin{bmatrix} 1 & 1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix} = 4v.$$

The corresponding eigenvalue is $\lambda = 4$.

Remark

Note that if $A v = \lambda v$ and $c$ is any scalar, then

$$A (c v) = c \ A v = c (\lambda v) = \lambda (c v).$$

Consequently, if $v$ is an eigenvector of $A$, then so is $c v$ for any nonzero scalar $c$. 
The eigenvector/eigenvalue equation can be rewritten as

$$(A - \lambda I) \mathbf{v} = \mathbf{0}.$$

The eigenvalues of $A$ are the values of $\lambda$ for which the above equation has nontrivial solutions. There are nontrivial solutions if and only if

$$\det (A - \lambda I) = 0.$$

**Definition**

For a given $n \times n$ matrix $A$, the polynomial

$$p(\lambda) = \det(A - \lambda I)$$

is called the **characteristic polynomial** of $A$, and the equation

$$p(\lambda) = 0$$

is called the **characteristic equation** of $A$.

The eigenvalues of $A$ are the roots of its characteristic polynomial.
Finding eigenvectors

If $\lambda$ is a root of the characteristic polynomial, then the nonzero elements of
\[
\text{nullspace } (A - \lambda I)
\]
will be eigenvectors for $A$.

Since nonzero linear combinations of eigenvectors for a single eigenvalue are still eigenvectors, we’ll find a set of linearly independent eigenvectors for each eigenvalue.
Find all of the eigenvalues and eigenvectors of

\[ A = \begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix}. \]

Compute the characteristic polynomial

\[
\det(A - \lambda I) = \begin{vmatrix} 5 - \lambda & -4 \\ 8 & -7 - \lambda \end{vmatrix} = \lambda^2 + 2\lambda - 3.
\]

Its roots are \( \lambda = -3 \) and \( \lambda = 1 \). These are the eigenvalues. If \( \lambda = -3 \), we have the eigenvector \((1, 2)\).

If \( \lambda = 1 \), then

\[ A - I = \begin{bmatrix} 4 & -4 \\ 8 & -8 \end{bmatrix}, \]

which gives us the eigenvector \((1, 1)\).
Find all of the eigenvalues and eigenvectors of

\[
A = \begin{bmatrix}
5 & 12 & -6 \\
-3 & -10 & 6 \\
-3 & -12 & 8
\end{bmatrix}.
\]

Compute the characteristic polynomial \(-(\lambda - 2)^2(\lambda + 1)\).

**Definition**

If \(A\) is a matrix with characteristic polynomial \(p(\lambda)\), the multiplicity of a root \(\lambda\) of \(p\) is called the **algebraic multiplicity** of the eigenvalue \(\lambda\).

**Example**

In the example above, the eigenvalue \(\lambda = 2\) has algebraic multiplicity 2, while \(\lambda = -1\) has algebraic multiplicity 1.
The eigenvalue $\lambda = 2$ gives us two linearly independent eigenvectors $(-4, 1, 0)$ and $(2, 0, 1)$.

When $\lambda = -1$, we obtain the single eigenvector $(-1, 1, 1)$.

**Definition**
The number of linearly independent eigenvectors corresponding to a single eigenvalue is its **geometric multiplicity**.

**Example**
Above, the eigenvalue $\lambda = 2$ has geometric multiplicity 2, while $\lambda = -1$ has geometric multiplicity 1.

**Theorem**
The geometric multiplicity of an eigenvalue is less than or equal to its algebraic multiplicity.

**Definition**
A matrix that has an eigenvalue whose geometric multiplicity is less than its algebraic multiplicity is called **defective**.
Find all of the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$  

The characteristic polynomial is \((\lambda - 1)^2\), so we have a single eigenvalue \(\lambda = 1\) with algebraic multiplicity 2. The matrix

$$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has a one-dimensional null space spanned by the vector \((1, 0)\). Thus, the geometric multiplicity of this eigenvalue is 1.
Find all of the eigenvalues and eigenvectors of

\[ A = \begin{bmatrix} -2 & -6 \\ 3 & 4 \end{bmatrix}. \]

The characteristic polynomial is \( \lambda^2 - 2\lambda + 10 \). Its roots are

\[ \lambda_1 = 1 + 3i \quad \text{and} \quad \lambda_2 = \overline{\lambda_1} = 1 - 3i. \]

The eigenvector corresponding to \( \lambda_1 \) is \((-1 + i, 1)\).

**Theorem**

Let \( A \) be a square matrix with real elements. If \( \lambda \) is a complex eigenvalue of \( A \) with eigenvector \( \mathbf{v} \), then \( \overline{\lambda} \) is an eigenvalue of \( A \) with eigenvector \( \overline{\mathbf{v}} \).

**Example**

The eigenvector corresponding to \( \lambda_2 = \overline{\lambda_1} \) is \((-1 - i, 1)\).
If an $n \times n$ matrix $A$ is nondefective, then a set of linearly independent eigenvectors for $A$ will form a basis for $\mathbb{R}^n$. If we express the linear transformation $T(x) = Ax$ as a matrix transformation relative to this basis, it will look like

$$
\begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2 \\
\vdots & \ddots \\
0 & \cdots & \lambda_n
\end{bmatrix}.
$$

The following example will demonstrate the utility of such a representation.
Determine all solutions to the linear system of differential equations

\[ \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} 5x_1 - 4x_2 \\ 8x_1 - 7x_2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax. \]

We know that the coefficient matrix has eigenvalues \( \lambda_1 = 1 \) and \( \lambda_2 = -3 \) with corresponding eigenvectors \( v_1 = (1, 1) \) and \( v_2 = (1, 2) \), respectively. Using the basis \( \{v_1, v_2\} \), we write the linear transformation \( T(x) = Ax \) in the matrix representation

\[ \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}. \]
Now consider the new linear system

\[
\begin{bmatrix}
    y_1' \\
    y_2'
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = By.
\]

It has the obvious solution

\[
y_1 = c_1 e^t \quad \text{and} \quad y_2 = c_2 e^{-3t},
\]

for any scalars \(c_1\) and \(c_2\). How is this relevant to \(x' = Ax\)?

\[
A \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} Av_1 & Av_2 \end{bmatrix} = \begin{bmatrix} v_1 & -3v_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} B.
\]

Let \(S = \begin{bmatrix} v_1 & v_2 \end{bmatrix}\). Since \(y' = By\) and \(AS = SB\), we have

\[
(Sy)' = Sy' = SBy = ASy = A (Sy).
\]

Thus, a solution to \(x' = Ax\) is given by

\[
x = Sy = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 e^t \\ c_2 e^{-3t} \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 e^{-3t} \\ c_1 e^t + 2c_2 e^{-3t} \end{bmatrix}.
\]