Solving Linear Systems, Continued and The Inverse of a Matrix

Math 240 — Calculus III
Summer 2015, Session II
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1. Solving Linear Systems
   Gauss-Jordan elimination
   The rank of a matrix

2. The inverse of a square matrix
   Definition
   Computing inverses
   Properties of inverses
   Using inverse matrices
   Conclusion
**Gaussian elimination** solves a linear system by reducing to REF via elementary row ops and then using back substitution.

**Example**

\[
\begin{align*}
3x_1 - 2x_2 + 2x_3 &= 9 \\
x_1 - 2x_2 + x_3 &= 5 \\
2x_1 - x_2 - 2x_3 &= -1
\end{align*}
\]

\[
\begin{bmatrix}
3 & -2 & 2 & 9 \\
1 & -2 & 1 & 5 \\
2 & -1 & -2 & -1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -2 & 1 & 5 \\
0 & 1 & 3 & 5 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

Back substitution gives the solution \((1, -1, 2)\).
Reducing the augmented matrix to RREF makes the system even easier to solve.

**Example**

\[
\begin{bmatrix}
1 & -2 & 1 & 5 \\
0 & 1 & 3 & 5 \\
0 & 0 & 1 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

\[\begin{aligned}
x_1 &= 1 \\
x_2 &= -1 \\
x_3 &= 2
\end{aligned}\]

**Steps**

1. \( A_{32}(-3) \)
2. \( A_{31}(-1) \)
3. \( A_{21}(2) \)

Now, without any back substitution, we can see that the solution is \((1, -1, 2)\).

The method of solving a linear system by reducing its augmented matrix to RREF is called **Gauss-Jordan elimination**.
The rank of a matrix

Definition
The **rank** of a matrix, $A$, is the number of nonzero rows it has after reduction to REF. It is denoted by $\text{rank}(A)$.

If $A$ is the coefficient matrix of an $m \times n$ linear system and $\text{rank}(A^\#) = \text{rank}(A) = n$ then the REF looks like

$$
\begin{bmatrix}
1 & * & * & \cdots & * \\
1 & * & \cdots & * \\
0 & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & * \\
0 & \cdots & \cdots & 0
\end{bmatrix} \xrightarrow{\sim}
\begin{align*}
x_1 &= * \\
x_2 &= * \\
&\vdots \\
x_n &= * 
\end{align*}
$$

Lemma
*Suppose* $Ax = b$ *is an* $m \times n$ *linear system with augmented matrix* $A^\#$.* If* $\text{rank}(A^\#) = \text{rank}(A) = n$ *then the system has a unique solution.*
Example

Determine the solution set of the linear system

\[
\begin{align*}
    x_1 + x_2 - x_3 + x_4 &= 1, \\
    2x_1 + 3x_2 + x_3 &= 4, \\
    3x_1 + 5x_2 + 3x_3 - x_4 &= 5.
\end{align*}
\]

Reduce the augmented matrix.

\[
\begin{bmatrix}
    1 & 1 & -1 & 1 & 1 \\
    2 & 3 & 1 & 0 & 4 \\
    3 & 5 & 3 & -1 & 5
\end{bmatrix}
\begin{bmatrix}
    A_{12}(-2) \\
    A_{13}(-3) \\
    A_{23}(-2)
\end{bmatrix}
\rightarrow
\begin{bmatrix}
    1 & 1 & -1 & 1 & 1 \\
    0 & 1 & 3 & -2 & 2 \\
    0 & 0 & 0 & 0 & -2
\end{bmatrix}
\]

The last row says \(0 = -2\); the system is inconsistent.

Lemma

Suppose \(Ax = b\) is a linear system with augmented matrix \(A\#\). If \(\text{rank}(A\#) > \text{rank}(A)\) then the system is inconsistent.
Example

Determine the solution set of the linear system

\[
\begin{align*}
5x_1 - 6x_2 + x_3 &= 4, \\
2x_1 - 3x_2 + x_3 &= 1, \\
4x_1 - 3x_2 - x_3 &= 5.
\end{align*}
\]

Reduce the augmented matrix.

\[
\begin{bmatrix}
5 & -6 & 1 & 4 \\
2 & -3 & 1 & 1 \\
4 & -3 & -1 & 5
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -1 & 2 \\
0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
x_1 & - x_3 = 2 \\
x_2 & - x_3 = 1
\end{bmatrix}
\]

The unknown \(x_3\) can assume any value. Let \(x_3 = t\). Then by back substitution we get \(x_2 = t + 1\) and \(x_1 = t + 2\). Thus, the solution set is the line

\[
\{(t + 2, t + 1, t) : t \in \mathbb{R}\}.
\]
Definition

When an unknown variable in a linear system is free to assume any value, we call it a **free variable**. Variables that are not free are called **bound variables**.

The value of a bound variable is uniquely determined by a choice of values for all of the free variables in the system.

Lemma

Suppose $Ax = b$ is an $m \times n$ linear system with augmented matrix $A^\#$. If $\text{rank}(A^\#) = \text{rank}(A) < n$ then the system has an infinite number of solutions. Such a system will have $n - \text{rank}(A)$ free variables.
Example

Use Gaussian elimination to solve

\[
\begin{align*}
    x_1 + 2x_2 - 2x_3 - x_4 &= 3, \\
    3x_1 + 6x_2 + x_3 + 11x_4 &= 16, \\
    2x_1 + 4x_2 - x_3 + 4x_4 &= 9.
\end{align*}
\]

Reducing to row-echelon form yields

\[
\begin{align*}
    x_1 + 2x_2 - 2x_3 - x_4 &= 3, \\
    x_3 + 2x_4 &= 1.
\end{align*}
\]

Choose as free variables those variables that do not have a pivot in their column.

In this case, our free variables will be \(x_2\) and \(x_4\). The solution set is the plane

\[
\{(5 - 2s - 3t, s, 1 - 2t, t) : s, t \in \mathbb{R}\}.
\]
The inverse of a square matrix

Can we divide by a matrix? What properties should the inverse matrix have?

Definition
Suppose $A$ is a square, $n \times n$ matrix. An inverse matrix for $A$ is an $n \times n$ matrix, $B$, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$  

If $A$ has such an inverse then we say that it is invertible or nonsingular. Otherwise, we say that $A$ is singular.

Remark
Not every matrix is invertible.

If you have a linear system $Ax = b$ and $B$ is an inverse matrix for $A$ then the linear system has the unique solution

$$x = Bb.$$  

Example

If

\[
A = \begin{bmatrix}
1 & -1 & 2 \\
2 & -3 & 3 \\
1 & -1 & 1
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
0 & -1 & 3 \\
1 & -1 & 1 \\
1 & 0 & -1
\end{bmatrix}
\]

then \( B \) is the inverse of \( A \).

Theorem (Matrix inverses are well-defined)

Suppose \( A \) is an \( n \times n \) matrix. If \( B \) and \( C \) are two inverses of \( A \) then \( B = C \).

Thus, we can write \( A^{-1} \) for the inverse of \( A \) with no ambiguity.

Useful Example

If \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( ad - bc \neq 0 \) then \( A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \).
Inverse matrices sound great! How do I find one?
Suppose \( A \) is a \( 3 \times 3 \) invertible matrix. If \( A^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \) then
\[
\begin{align*}
A x_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \\
A x_2 &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} , \\
A x_3 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .
\end{align*}
\]

We can find \( A^{-1} \) by solving 3 linear systems at once!

In general, form the augmented matrix and reduce to RREF. You end up with \( A^{-1} \) on the right.
\[
\begin{bmatrix} A | I_n \end{bmatrix} \sim \begin{bmatrix} I_n | A^{-1} \end{bmatrix}
\]
Finding the inverse of a matrix

Example

Let’s find the inverse of \( A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix} \).

Take the augmented matrix and row reduce.

\[
\begin{bmatrix}
1 & -1 & 2 & | & 1 & 0 & 0 \\
2 & -3 & 3 & | & 0 & 1 & 0 \\
1 & -1 & 1 & | & 0 & 0 & 1 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & | & 0 & -1 & 3 \\
0 & 1 & 0 & | & 1 & -1 & 1 \\
0 & 0 & 1 & | & 1 & 0 & -1 \\
\end{bmatrix}
\]

\( A^{-1} \)

Steps

1. \( A_{12}(-2) \)
2. \( A_{13}(-1) \)
3. \( M_2(-1) \)
4. \( M_3(-1) \)
5. \( A_{32}(-1) \)
6. \( A_{31}(-2) \)
7. \( A_{21}(1) \)
Finding the inverse of a matrix

In order to find the inverse of a matrix, $A$, we row reduced an augmented matrix with $A$ on the left. What if we don’t end up with $I_n$ on the left?

**Theorem**

An $n \times n$ matrix, $A$, is invertible if and only if $\text{rank}(A) = n$.

**Example**

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$.

Try to reduce the matrix to RREF.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \xrightarrow{A_{12}(-2)} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

Since $\text{rank}(A) < 2$, we conclude that $A$ is not invertible.

Notice that $(1)(6) - (3)(2) = 0$. 
Finding the inverse of a matrix

Diagonal matrices have simple inverses.

**Proposition**

*The inverse of a diagonal matrix is the diagonal matrix with reciprocal entries.*

\[
\begin{bmatrix}
a_{11} & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \cdots & a_{nn} & 0 \\
\end{bmatrix}^{-1} =
\begin{bmatrix}
a_{11}^{-1} & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \cdots & a_{nn}^{-1} & 0 \\
\end{bmatrix}
\]

Upper and lower triangular matrices have inverses of the same form.

**Proposition**

*The inverse of an upper triangular matrix is upper triangular. The inverse of a lower triangular matrix is lower triangular.*
Properties of inverse matrices

Suppose \( A \) and \( B \) are \( n \times n \) invertible matrices.

- \( A^{-1} \) is invertible and \((A^{-1})^{-1} = A\).
- \( AB \) is invertible and \((AB)^{-1} = B^{-1}A^{-1}\).
- \( A^T \) is invertible and \((A^T)^{-1} = (A^{-1})^T\).

Corollary

Suppose \( A_1, A_2, \ldots, A_k \) are invertible \( n \times n \) matrices. Then their product, \( A_1A_2\cdots A_k \) is invertible, and

\[
(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_1^{-1}.
\]
Recall that if $A$ is an invertible matrix then the linear system $Ax = b$ has the unique solution $x = A^{-1}b$.

**Example**

Solve the linear system

\[
\begin{align*}
x_1 + 3x_2 &= 1, \\
2x_1 + 5x_2 &= 3.
\end{align*}
\]

The coefficient matrix is $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, so $A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$.

The inverse of a $2 \times 2$ matrix is

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{when } ad - bc \neq 0.
\]

Hence,

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.
\]
Inverse matrices are an elegant way of solving linear systems. They do have some drawbacks:

- They are only applicable when the coefficient matrix is square.
- Even in the case of a square matrix, an inverse may not exist.
- They are hard to compute, at least as complicated as doing Gauss-Jordan elimination.

However, they can be useful if

- the coefficient matrix has an obvious inverse,
- you need to solve multiple linear systems with the same coefficients.