The Determinant

Math 240 — Calculus III

Summer 2015, Session II

Wednesday, July 8, 2015
1. Definition of the determinant

2. Computing determinants

3. Properties of determinants
Yesterday: \( Ax = b \) has a unique solution when \( A \) is square and nonsingular.

Today: how to determine whether \( A \) is invertible.

Example

Recall that a 2 \( \times \) 2 matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is invertible as long as \( ad - bc \neq 0 \). The quantity \( ad - bc \) is the determinant of this matrix and the matrix is invertible exactly when its determinant is nonzero.
What should the determinant be?

- We want to associate a number with a matrix that is zero if and only if the matrix is singular.
- An $n \times n$ matrix is nonsingular if and only if its rank is $n$.
- For upper triangular matrices, the rank is the number of nonzero entries on the diagonal.
- To determine if every number in a set is nonzero, we can multiply them.

**Definition**

The **determinant** of an upper triangular matrix, $A = [a_{ij}]$, is the product of the elements $a_{ii}$ along its main diagonal. We write

$$
\text{det}(A) = \begin{vmatrix}
    a_{11} & \cdots & a_{1n} \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & a_{nn}
\end{vmatrix} = a_{11}a_{22}\cdots a_{nn}.
$$
What should the determinant be?

What about matrices that are not upper triangular? We can make any matrix upper triangular via row reduction. So how do elementary row operations affect the determinant?

- $M_i(k)$ multiplies the determinant by $k$. (Remember that $k$ cannot be zero.)
- $A_{ij}(k)$ does not change the determinant.
- $P_{ij}$ multiplies the determinant by $-1$.

Let’s extend these properties to all matrices.

Definition

The **determinant** of a square matrix, $A$, is the determinant of any upper triangular matrix obtained from $A$ by row reduction times $\frac{1}{k}$ for every $M_i(k)$ operation used while reducing as well as $-1$ for each $P_{ij}$ operation used.
Example

Compute $\det(A)$, where $A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 3 & 10 \\ 1 & -1 & 0 \end{bmatrix}$.

We need to put $A$ in upper triangular form.

So the determinant is

$$\begin{vmatrix} 0 & 2 & 1 \\ 2 & 3 & 10 \\ 1 & -1 & 0 \end{vmatrix} = (-1)(5) \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{vmatrix} = 15.$$
Important Example

Given a general $2 \times 2$ matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, compute $\det(A)$.

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{A_{12}(-c/a)} \begin{bmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{bmatrix}
\]

so

\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{vmatrix} = ad - bc.
\]

This explains

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

when $ad - bc \neq 0$. 
Other methods of computing determinants

Theorem (Cofactor expansion)

Suppose $A = [a_{ij}]$ is an $n \times n$ matrix. For any fixed $k$ between 1 and $n$,

$$
\det(A) = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det(A_{kj}) = \sum_{i=1}^{n} (-1)^{i+k} a_{ik} \det(A_{ik})
$$

where $A_{ij}$ is the $(n - 1) \times (n - 1)$ submatrix obtained by removing the $i^{th}$ row and $j^{th}$ column from $A$.

Example

$$
\begin{vmatrix}
   i & j & k \\
   a & b & c \\
   d & e & f
\end{vmatrix} = \begin{vmatrix}
   b & c \\
   e & f
\end{vmatrix} i - \begin{vmatrix}
   a & c \\
   d & f
\end{vmatrix} j + \begin{vmatrix}
   a & b \\
   d & e
\end{vmatrix} k.
$$
Corollary

If $A = [a_{i,j}]$ is an $n \times n$ matrix and the element $a_{i,j}$ is the only nonzero entry in its row or column then

$$\det(A) = (-1)^{i+j} a_{i,j} \det(A_{i,j}).$$

Example

$$\begin{vmatrix} 0 & 2 & 1 \\ 3 & 0 & 0 \\ 0 & 1 & 5 \end{vmatrix} = -3 \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} = -27.$$
Some of you may have learned the method of computing a $3 \times 3$ determinant by multiplying diagonals.

Be aware that this method does not work for matrices larger than $3 \times 3$. 
Properties of determinants

Theorem (Main theorem)
Suppose $A$ is an $n \times n$ matrix. The following are equivalent:

- $A$ is invertible,
- $\det(A) \neq 0$.

Further properties

- $\det(A^T) = \det(A)$.
- The determinant of a lower triangular matrix is also the product of the elements on the main diagonal.
- If $A$ has a row or column of zeros then $\det(A) = 0$.
- If two rows or columns of $A$ are the same then $\det(A) = 0$.
- $\det(AB) = \det(A) \det(B)$, $\det(A^{-1}) = \det(A)^{-1}$.
- $\det(sA) = s^n \det(A)$.
- It is not true that $\det(A + B) = \det(A) + \det(B)$.
Let $A$ be an $n \times n$ matrix and $a_1, \ldots, a_n$ be the rows or columns of $A$.

**Theorem**

The volume (or area, if $n = 2$) of the parallelepiped determined by the vectors $a_1, \ldots, a_n$ is $|\det(A)|$.

**Corollary**

The vectors $a_1, \ldots, a_n$ lie in the same hyperplane if and only if $\det(A) = 0$. 

Source: en.wikibooks.org/wiki/Linear_Algebra