Eigenvalues, Eigenvectors, and Diagonalization

Math 240 — Calculus III

Summer 2015, Session II

Thursday, July 16, 2015
1. Eigenvalues and Eigenvectors

2. Diagonalization
Next week, we will apply linear algebra to solving differential equations. One that is particularly easy to solve is

\[ y' = ay. \]

It has the solution \( y = ce^{at} \), where \( c \) is any real (or complex) number. Viewed in terms of linear transformations, \( y = ce^{at} \) is the solution to the vector equation

\[ T(y) = ay, \tag{1} \]

where \( T : C^k(I) \rightarrow C^{k-1}(I) \) is \( T(y) = y' \). We are going to study equation (1) in a more general context.
Definition

Let $A$ be an $n \times n$ matrix. Any value of $\lambda$ for which

$$Av = \lambda v$$

has nontrivial solutions $v$ are called eigenvalues of $A$. The corresponding nonzero vectors $v$ are called eigenvectors of $A$.

Figure: A geometrical description of eigenvectors in $\mathbb{R}^2$. 
Example

If $A$ is the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -3 & 5 \end{bmatrix},$$

then the vector $v = (1, 3)$ is an eigenvector for $A$ because

$$Av = \begin{bmatrix} 1 & 1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix} = 4v.$$

The corresponding eigenvalue is $\lambda = 4$.

Remark

Note that if $Av = \lambda v$ and $c$ is any scalar, then

$$A(cv) = c Av = c(\lambda v) = \lambda(cv).$$

Consequently, if $v$ is an eigenvector of $A$, then so is $cv$ for any nonzero scalar $c$. 
Finding eigenvalues

The eigenvector/eigenvalue equation can be rewritten as

\[(A - \lambda I)\mathbf{v} = \mathbf{0}.\]

The eigenvalues of \(A\) are the values of \(\lambda\) for which the above equation has nontrivial solutions. There are nontrivial solutions if and only if

\[\det (A - \lambda I) = 0.\]

**Definition**

For a given \(n \times n\) matrix \(A\), the polynomial

\[p(\lambda) = \det (A - \lambda I)\]

is called the **characteristic polynomial** of \(A\), and the equation

\[p(\lambda) = 0\]

is called the **characteristic equation** of \(A\).

The eigenvalues of \(A\) are the roots of its characteristic polynomial.
If $\lambda$ is a root of the characteristic polynomial, then the nonzero elements of

$$\text{nullspace } (A - \lambda I)$$

will be eigenvectors for $A$.

Since nonzero linear combinations of eigenvectors for a single eigenvalue are still eigenvectors, we’ll find a set of linearly independent eigenvectors for each eigenvalue.
Find all of the eigenvalues and eigenvectors of

\[ A = \begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix}. \]

Compute the characteristic polynomial

\[
\det(A - \lambda I) = \begin{vmatrix} 5 - \lambda & -4 \\ 8 & -7 - \lambda \end{vmatrix} = \lambda^2 + 2\lambda - 3.
\]

Its roots are \( \lambda = -3 \) and \( \lambda = 1 \). These are the eigenvalues.

If \( \lambda = -3 \), we have the eigenvector \((1, 2)\).

If \( \lambda = 1 \), then

\[ A - I = \begin{bmatrix} 4 & -4 \\ 8 & -8 \end{bmatrix}, \]

which gives us the eigenvector \((1, 1)\).
Find all of the eigenvalues and eigenvectors of

\[
A = \begin{bmatrix}
5 & 12 & -6 \\
-3 & -10 & 6 \\
-3 & -12 & 8
\end{bmatrix}.
\]

Compute the characteristic polynomial \(- (\lambda - 2)^2 (\lambda + 1)\).

**Definition**

If \( A \) is a matrix with characteristic polynomial \( p(\lambda) \), the multiplicity of a root \( \lambda \) of \( p \) is called the **algebraic multiplicity** of the eigenvalue \( \lambda \).

**Example**

In the example above, the eigenvalue \( \lambda = 2 \) has algebraic multiplicity 2, while \( \lambda = -1 \) has algebraic multiplicity 1.
The eigenvalue $\lambda = 2$ gives us two linearly independent eigenvectors $(-4, 1, 0)$ and $(2, 0, 1)$. When $\lambda = -1$, we obtain the single eigenvector $(-1, 1, 1)$.

**Definition**
The number of linearly independent eigenvectors corresponding to a single eigenvalue is its **geometric multiplicity**.

**Example**
Above, the eigenvalue $\lambda = 2$ has geometric multiplicity 2, while $\lambda = -1$ has geometric multiplicity 1.

**Theorem**
*The geometric multiplicity of an eigenvalue is less than or equal to its algebraic multiplicity.*

**Definition**
A matrix that has an eigenvalue whose geometric multiplicity is less than its algebraic multiplicity is called **defective**.
Find all of the eigenvalues and eigenvectors of

\[ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \]

The characteristic polynomial is \((\lambda - 1)^2\), so we have a single eigenvalue \(\lambda = 1\) with algebraic multiplicity 2. The matrix

\[ A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]

has a one-dimensional null space spanned by the vector \((1, 0)\). Thus, the geometric multiplicity of this eigenvalue is 1.
Find all of the eigenvalues and eigenvectors of

\[ A = \begin{bmatrix} -2 & -6 \\ 3 & 4 \end{bmatrix}. \]

The characteristic polynomial is \( \lambda^2 - 2\lambda + 10 \). Its roots are

\[ \lambda_1 = 1 + 3i \quad \text{and} \quad \lambda_2 = \overline{\lambda_1} = 1 - 3i. \]

The eigenvector corresponding to \( \lambda_1 \) is \((-1 + i, 1)\).

**Theorem**

Let \( A \) be a square matrix with real elements. If \( \lambda \) is a complex eigenvalue of \( A \) with eigenvector \( \mathbf{v} \), then \( \overline{\lambda} \) is an eigenvalue of \( A \) with eigenvector \( \overline{\mathbf{v}} \).

**Example**

The eigenvector corresponding to \( \lambda_2 = \overline{\lambda_1} \) is \((-1 - i, 1)\).
If an \( n \times n \) matrix \( A \) is nondefective, then a set of linearly independent eigenvectors for \( A \) will form a basis for \( \mathbb{R}^n \). If we express the linear transformation \( T(x) = Ax \) as a matrix transformation relative to this basis, it will look like

\[
\begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2 \\
& \ddots \\
0 & \cdots & \cdots & \lambda_n
\end{bmatrix}.
\]

The following example will demonstrate the utility of such a representation.
Determine all solutions to the linear system of differential equations

\[
x' = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} 5x_1 - 4x_2 \\ 8x_1 - 7x_2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax.
\]

We know that the coefficient matrix has eigenvalues \( \lambda_1 = 1 \) and \( \lambda_2 = -3 \) with corresponding eigenvectors \( v_1 = (1, 1) \) and \( v_2 = (1, 2) \), respectively. Using the basis \( \{v_1, v_2\} \), we write the linear transformation \( T(x) = Ax \) in the matrix representation

\[
\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}.
\]
Now consider the new linear system

\[ y' = \begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = By. \]

It has the obvious solution

\[ y_1 = c_1 e^t \quad \text{and} \quad y_2 = c_2 e^{-3t}, \]

for any scalars \( c_1 \) and \( c_2 \). How is this relevant to \( x' = Ax \)?

\[ A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} Av_1 \\ Av_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -3v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} B. \]

Let \( S = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \). Since \( y' = By \) and \( AS = SB \), we have

\[ (Sy)' = Sy' = SBy = ASy = A (Sy). \]

Thus, a solution to \( x' = Ax \) is given by

\[ x = Sy = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 e^t \\ c_2 e^{-3t} \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 e^{-3t} \\ c_1 e^t + 2c_2 e^{-3t} \end{bmatrix}. \]