Linear Systems of Differential Equations

Math 240 — Calculus III

Summer 2015, Session II

Tuesday, July 21, 2015
1. First order linear systems
   Solutions to vector differential equations
   Beyond first order systems
Definition

A first order system of differential equations is of the form

\[ x'(t) = A(t)x(t) + b(t), \]

where \( A(t) \) is an \( n \times n \) matrix function and \( x(t) \) and \( b(t) \) are \( n \)-vector functions. Also called a vector differential equation.

Example

The linear system

\[
\begin{align*}
x_1'(t) &= \cos(t)x_1(t) - \sin(t)x_2(t) + e^{-t} \\
x_2'(t) &= \sin(t)x_1(t) + \cos(t)x_2(t) - e^{-t}
\end{align*}
\]

can also be written as the vector differential equation

\[ x'(t) = A(t)x(t) + b(t) \]

where

\[
A(t) = \begin{bmatrix}
\cos(t) & -\sin(t) \\
\sin(t) & \cos(t)
\end{bmatrix}, \quad x(t) = \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}, \quad \text{and} \quad b(t) = \begin{bmatrix}
e^{-t} \\
-e^{-t}
\end{bmatrix}.
\]
The vector space $V_n(I)$

A solution to a vector differential equation will be an element of the vector space $V_n(I)$ consisting of column $n$-vector functions defined on the interval $I$.

**Definition**

Suppose $x_1(t), x_2(t), \ldots, x_n(t) \in V_n(I)$. The **Wronskian** of these vectors is

$$W[x_1, \ldots, x_n](t) = \begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ \vdots & \vdots & & \vdots \\ \end{vmatrix}.$$ 

**Theorem**

If $W[x_1, \ldots, x_n](t)$ is nonzero for at least one $t \in I$, then \{x_1(t), \ldots, x_n(t)\} is a linearly independent subset of $V_n(I)$. 

As with linear systems, a homogeneous linear system of differential equations is one in which $b(t) = 0$.

**Theorem**

If $A(t)$ is an $n \times n$ matrix function that is continuous on the interval $I$, then the set of all solutions to $x'(t) = A(t)x(t)$ is a subspace of $V_n(I)$ of dimension $n$.

**Proof.**

Up to you. Proof of dim = $n$ later, if there’s time.  \[Q.E.D.\]
The general solution: homogeneous case

If the solution set is a vector space of dimension $n$, it has a basis.

**Definition**

Any set $\{x_1, x_2, \ldots, x_n\}$ of $n$ solutions to $x' = Ax$ that is linearly independent on $I$ is called a **fundamental set of solutions** on $I$. Any solution may be written in the form

$$x(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t),$$

which is called the **general solution**.

**Theorem**

If $A(t)$ is an $n \times n$ matrix function that is continuous on an interval $I$, and $\{x_1, x_2, \ldots, x_n\}$ is a linearly independent set of solutions to $x' = Ax$ on $I$, then

$$W[x_1, x_2, \ldots, x_n](t) \neq 0$$

for every $t \in I$. 
The general solution: nonhomogeneous case

The case of nonhomogeneous systems is also familiar.

**Theorem**

Suppose $A(t)$ is an $n \times n$ matrix function continuous on an interval $I$ and \{${\bf x}_1, \ldots, {\bf x}_n$\} is a fundamental set of solutions to the equation $x'(t) = A(t)x(t)$. If $x = x_p(t)$ is any particular solution to the nonhomogeneous vector differential equation

$$x'(t) = A(t)x(t) + b(t)$$

on $I$, then every solution to this equation on $I$ is in the form of the **general solution**

$$x'(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t) + x_p(t),$$

where $x_p(t)$ is any particular solution.

The two pieces of the general solution are the **particular solution**, $x_p(t)$, and the **complementary solution**, $x_c(t)$. 
Sometimes, we are interested in one particular solution to a vector differential equation.

**Definition**

An **initial value problem** consists of a vector differential equation

$$x'(t) = A(t)x(t) + b(t)$$

and an **initial condition**

$$x(t_0) = x_0$$

with known, fixed values for $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$.

**Theorem**

When $A(t)$ and $b(t)$ are continuous on an interval $I$, the above initial value problem has a unique solution on $I$. 
Turning higher order linear systems into first order

Aren’t we a little limited if all we can solve are first order differential equations? Not always.

**Example**

Consider the linear *second* order system

\[
\begin{align*}
x''(t) - 4y(t) &= e^t, \\
y''(t) + x'(t) &= \sin t.
\end{align*}
\]

Introduce new variables

\[
x_1(t) = x(t), \quad x_2(t) = x'(t), \quad x_3(t) = y(t), \quad x_4(t) = y'(t).
\]

Then the above equations can be replaced with

\[
\begin{align*}
x_2'(t) - 4x_3(t) &= e^t, \\
x_4'(t) + x_2(t) &= \sin t,
\end{align*}
\]

and we must supplement them with the equations

\[
x_1'(t) = x_2(t), \quad x_3'(t) = x_4(t).
\]