Higher Order Linear Differential Equations

Math 240 — Calculus III

Summer 2015, Session II

Tuesday, July 28, 2015
1. Linear differential equations of order $n$
   Linear differential operators
   Familiar stuff
   An example

2. Homogeneous constant-coefficient linear differential equations
We now turn our attention to solving **linear differential equations of order** $n$. The general form of such an equation is

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x),$$

where $a_0, a_1, \ldots, a_n$, and $F$ are functions defined on an interval $I$.

The general strategy is to reformulate the above equation as

$$Ly = F,$$

where $L$ is an appropriate linear transformation. In fact, $L$ will be a **linear differential operator**.
Recall that the mapping $D : C^k(I) \to C^{k-1}(I)$ defined by $D(f) = f'$ is a linear transformation. This $D$ is called the \textbf{derivative operator}. Higher order derivative operators $D^k : C^k(I) \to C^0(I)$ are defined by composition:

$$D^k = D \circ D^{k-1},$$

so that

$$D^k(f) = \frac{d^k f}{dx^k}.$$

A \textbf{linear differential operator of order $n$} is a linear combination of derivative operators of order up to $n$,

$$L = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n,$$

defined by

$$Ly = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y,$$

where the $a_i$ are continuous functions of $x$. $L$ is then a linear transformation $L : C^n(I) \to C^0(I)$. (Why?)
Example

If \( L = D^2 + 4xD - 3x \), then

\[ Ly = y'' + 4xy' - 3xy. \]

We have

\[ L (\sin x) = -\sin x + 4x \cos x - 3x \sin x, \]
\[ L (x^2) = 2 + 8x^2 - 3x^3. \]

Example

If \( L = D^2 - e^{3x} D \), determine

1. \( L (2x - 3e^{2x}) = -12e^{2x} - 2e^{3x} + 6e^{5x} \)
2. \( L (3 \sin^2 x) = -3e^{3x} \sin 2x - 6 \cos 2x \)
Consider the general $n$-th order linear differential equation
\[ a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x), \]
where $a_0 \neq 0$ and $a_0, a_1, \ldots, a_n$, and $F$ are functions on an interval $I$.

If $a_0(x)$ is nonzero on $I$, then we may divide by it and relabel, obtaining
\[ y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x), \]
which we rewrite as
\[ Ly = F(x), \]
where $L = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$.

If $F(x)$ is identically zero on $I$, then the equation is \textbf{homogeneous}, otherwise it is \textbf{nonhomogeneous}.
If we have a homogeneous linear differential equation

\[ Ly = 0, \]

its solution set will coincide with \( \text{Ker}(L) \). In particular, the kernel of a linear transformation is a subspace of its domain.

**Theorem**

*The set of solutions to a linear differential equation of order \( n \) is a subspace of \( C^n(I) \). It is called the solution space. The dimension of the solutions space is \( n \).*

Being a vector space, the solution space has a basis \( \{y_1(x), y_2(x), \ldots, y_n(x)\} \) consisting of \( n \) solutions. Any element of the vector space can be written as a linear combination of basis vectors

\[ y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x). \]

This expression is called the **general solution**.
The Wronskian

We can use the Wronskian

\[ W[y_1, y_2, \ldots, y_n](x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y'_1(x) & y'_2(x) & \cdots & y'_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ y^{(n-1)}_1(x) & y^{(n-1)}_2(x) & \cdots & y^{(n-1)}_n(x) \end{vmatrix} \]

to determine whether a set of solutions is linearly independent.

**Theorem**

Let \( y_1, y_2, \ldots, y_n \) be solutions to the \( n \)-th order differential equation \( Ly = 0 \) whose coefficients are continuous on \( I \). If \( W[y_1, y_2, \ldots, y_n](x) = 0 \) at any single point \( x \in I \), then \( \{y_1, y_2, \ldots, y_n\} \) is linearly dependent.

To summarize, the vanishing or nonvanishing of the Wronskian on an interval completely characterizes the linear dependence or independence of a set of solutions to \( Ly = 0 \).
Example

Verify that \( y_1(x) = \cos 2x \) and \( y_2(x) = 3 - 6 \sin^2 x \) are solutions to the differential equation \( y'' + 4y = 0 \) on \( (-\infty, \infty) \).

Determine whether they are linearly independent on this interval.

\[
W[y_1, y_2](x) = \begin{vmatrix}
\cos 2x & 3 - 6 \sin^2 x \\
-2 \sin 2x & -12 \sin x \cos x \\
\end{vmatrix}
\]

\[
= -6 \sin 2x \cos 2x + 6 \sin 2x \cos 2x = 0
\]

They are linearly dependent. In fact, \( 3y_1 - y_2 = 0 \).
Consider the nonhomogeneous linear differential equation \( Ly = F \). The **associated homogeneous equation** is \( Ly = 0 \).

**Theorem**

Suppose \( \{y_1, y_2, \ldots, y_n\} \) are \( n \) linearly independent solutions to the \( n \)-th order equation \( Ly = 0 \) on an interval \( I \), and \( y = y_p \) is any particular solution to \( Ly = F \) on \( I \). Then every solution to \( Ly = F \) on \( I \) is of the form

\[
y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_p,
\]

for appropriate constants \( c_1, c_2, \ldots, c_n \).

This expression is the **general solution** to \( Ly = F \). The components of the general solution are

- the **complementary function**, \( y_c \), which is the general solution to the associated homogeneous equation,
- the **particular solution**, \( y_p \).
Theorem
If $y = u_p$ and $y = v_p$ are particular solutions to $Ly = f(x)$ and $Ly = g(x)$, respectively, then $y = u_p + v_p$ is a solution to $Ly = f(x) + g(x)$.

Proof.
We have $L(u_p + v_p) = L(u_p) + L(v_p) = f(x) + g(x)$. Q.E.D.
Example

Determine all solutions to the differential equation $y'' + y' - 6y = 0$ of the form $y(x) = e^{rx}$, where $r$ is a constant.

Substituting $y(x) = e^{rx}$ into the equation yields

$$e^{rx}(r^2 + r - 6) = r^2 e^{rx} + r e^{rx} - 6e^{rx} = 0.$$ 

Since $e^{rx} \neq 0$, we just need $(r + 3)(r - 2) = 0$. Hence, the two solutions of this form are

$$y_1(x) = e^{2x} \quad \text{and} \quad y_2(x) = e^{-3x}.$$ 

Could this be a basis for the solution space? Check linear independence. Yes! The general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-3x}.$$
Example

Determine the general solution to the differential equation

$$y'' + y' - 6y = 8e^{5x}.$$  

We know the complementary function,

$$y_c(x) = c_1e^{2x} + c_2e^{-3x}.$$  

For the particular solution, we might guess something of the form $y_p(x) = ce^{5x}$. What should $c$ be? We want

$$8e^{5x} = y_p'' + y_p' - 6y_p = (25c + 5c - 6c)e^{5x}.$$  

Cancel $e^{5x}$ and then solve $8 = 24c$ to find $c = \frac{1}{3}$.

The general solution is

$$y(x) = c_1e^{2x} + c_2e^{-3x} + \frac{1}{3}e^{5x}.$$  

We just found solutions to the linear differential equation
\[ y'' + y' - 6y = 0 \]
of the form \( y(x) = e^{rx} \). In fact, we found all solutions. This technique will often work. If \( y(x) = e^{rx} \) then
\[ y'(x) = re^{rx}, \quad y''(x) = r^2e^{rx}, \quad \ldots, \quad y^{(n)}(x) = r^ne^{rx}. \]
So if \( r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n = 0 \) then \( y(x) = e^{rx} \) is a solution to the linear differential equation
\[ y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0. \]
Let’s develop this approach more rigorously.
Consider the homogeneous linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

with \textit{constant coefficients} $a_i$. Expressed as a linear differential operator, the equation is $P(D)y = 0$, where

$$P(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n.$$ 

\textbf{Definition}

A linear differential operator with constant coefficients, such as $P(D)$, is called a \textbf{polynomial differential operator}. The polynomial

$$P(r) = r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n$$

is called the \textbf{auxiliary polynomial}, and the equation $P(r) = 0$ the \textbf{auxiliary equation}. 

The auxiliary polynomial
Example
The equation $y'' + y' - 6y = 0$ has auxiliary polynomial

$$P(r) = r^2 + r - 6.$$  

Examples
Give the auxiliary polynomials for the following equations.

1. $y'' + 2y' - 3y = 0$ \hspace{1cm} $r^2 + 2r - 3$
2. $(D^2 - 7D + 24)y = 0$ \hspace{1cm} $r^2 - 7r + 24$
3. $y''' - 2y'' - 4y' + 8y = 0$ \hspace{1cm} $r^3 - 2r^2 - 4r + 8$

The roots of the auxiliary polynomial will determine the solutions to the differential equation.
The key fact that will allow us to solve constant-coefficient linear differential equations is that polynomial differential operators commute.

**Theorem**

*If* $P(D)$ and $Q(D)$ *are polynomial differential operators, then*

$$P(D)Q(D) = Q(D)P(D).$$

**Proof.**

For our purposes, it will suffice to consider the case where $P$ and $Q$ are linear.

Commuting polynomial differential operators will allow us to turn a root of the auxiliary polynomial into a solution to the corresponding differential equation.
Linear polynomial differential operators

In our example,

\[ y'' + y' - 6y = 0, \]

with auxiliary polynomial

\[ P(r) = r^2 + r - 6, \]

the roots of \( P(r) \) are \( r = 2 \) and \( r = -3 \). An equivalent statement is that \( r - 2 \) and \( r + 3 \) are linear factors of \( P(r) \).

The functions \( y_1(x) = e^{2x} \) and \( y_2(x) = e^{-3x} \) are solutions to

\[ y'_1 - 2y_1 = 0 \quad \text{and} \quad y'_2 + 3y_2 = 0, \]

respectively.

**Theorem**

The general solution to the linear differential equation

\[ y' - ay = 0 \]

is \( y(x) = ce^{ax} \).
Theorem

Suppose $P(D)$ and $Q(D)$ are polynomial differential operators

$$P(D)y_1 = 0 = Q(D)y_2.$$  

If $L = P(D)Q(D)$, then

$$Ly_1 = 0 = Ly_2.$$  

Proof.

$$P(D)Q(D)y_2 = P(D)\left(Q(D)y_2\right) = P(D)0 = 0$$

$$P(D)Q(D)y_1 = Q(D)P(D)y_1$$

$$= Q(D)\left(P(D)y_1\right) = Q(D)0 = 0 \quad Q.E.D.$$  

Example

The theorem implies that, since

$$(D - 2)y_1 = 0 \quad \text{and} \quad (D + 3)y_2 = 0,$$  

the functions $y_1(x) = e^{2x}$ and $y_2(x) = e^{-3x}$ are solutions to

$$y'' + y' - 6y = (D^2 + D - 6)y = (D - 2)(D + 3)y = 0.$$
Furthermore, solutions produced from different roots of the auxiliary polynomial are independent.

Example
If $y_1(x) = e^{2x}$ and $y_2(x) = e^{-3x}$, then

$$W[y_1, y_2](x) = \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix}$$

$$= e^{-x} \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5e^{-x} \neq 0.$$
If we can factor the auxiliary polynomial into distinct linear factors, then the solutions from each linear factor will combine to form a fundamental set of solutions.

**Example**

Determine the general solution to $y'' - y' - 2y = 0$.

The auxiliary polynomial is

$$P(r) = r^2 - r - 2 = (r - 2)(r + 1).$$

Its roots are $r_1 = 2$ and $r_2 = -1$. The functions $y_1(x) = e^{2x}$ and $y_2(x) = e^{-x}$ satisfy

$$(D - 2)y_1 = 0 = (D + 1)y_2.$$

Therefore, $y_1$ and $y_2$ are solutions to the original equation. Since we have 2 solutions to a $2^{nd}$ degree equation, they constitute a fundamental set of solutions; the general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-x}.$$
What can go wrong with this process? The auxiliary polynomial could have a multiple root. In this case, we would get one solution from that root, but not enough to form the general solution. Fortunately, there are more.

**Theorem**

The differential equation $\left(D - r\right)^{m}y = 0$ has the following $m$ linearly independent solutions:

$$e^{rx}, \ x e^{rx}, \ x^{2} e^{rx}, \ldots, \ x^{m-1} e^{rx}.$$  

**Proof.**

Check it.  

\[ Q.E.D. \]
Example

Determine the general solution to $y'' + 4y' + 4y = 0$.

1. The auxiliary polynomial is $r^2 + 4r + 4$.
2. It has the multiple root $r = -2$.
3. Therefore, two linearly independent solutions are $y_1(x) = e^{-2x}$ and $y_2(x) = xe^{-2x}$.
4. The general solution is $y(x) = e^{-2x}(c_1 + c_2x)$. 
What happens if the auxiliary polynomial has complex roots? Can we recover real solutions? Yes!

**Theorem**

If $P(D)y = 0$ is a linear differential equation with real constant coefficients and $(D - r)^m$ is a factor of $P(D)$ with $r = a + bi$ and $b \neq 0$, then

1. $P(D)$ must also have the factor $(D - \bar{r})^m$,
2. this factor contributes the complex solutions
   $$e^{(a \pm bi)x}, \quad xe^{(a \pm bi)x}, \quad \ldots, \quad x^{m-1}e^{(a \pm bi)x},$$
3. the real and imaginary parts of the complex solutions are linearly independent real solutions
   $$x^k e^{ax} \cos bx \quad \text{and} \quad x^k e^{ax} \sin bx$$

for $k = 0, 1, \ldots, m - 1$. 
Example

Determine the general solution to $y'' + 6y' + 25y = 0$.

1. The auxiliary polynomial is $r^2 + 6r + 25$.
2. Its has roots $r = -3 \pm 4i$.
3. Two independent real-valued solutions are
   $$y_1(x) = e^{-3x} \cos 4x \quad \text{and} \quad y_2(x) = e^{-3x} \sin 4x.$$
4. The general solution is
   $$y(x) = e^{-3x}(c_1 \cos 4x + c_2 \sin 4x).$$